INTEGRAL TECHNIQUES IN THE STUDY OF THE ORDINARY RENEWAL PROCESS AND ITS EXTENSIONS

By

KONSTANTINOS GAKIS

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IMPORTANT SYMBOLS AND NOTATION

In the text the following notation will be used to characterize the ordinary renewal process $\{X_i, i = 1, 2, \ldots\}$ and the corresponding ordinary renewal counting process $\{N(t), t \geq 0\}$ under consideration:

- $f(\cdot)$ and $F(\cdot)$ to denote respectively the probability density function and the distribution function of the inter-renewal times
- $f^{*(n)}(\cdot)$ to denote the n-fold convolution of $f(\cdot)$ with itself.
- $F^{(n)}(\cdot)$ to denote the integral of $f^{*(n)}(\cdot)$.
- μ to denote the mean inter-renewal time
- $m(\cdot)$ and $M(\cdot)$ to denote respectively the renewal density and the renewal function
- $\mu^{(2)}$ and $\mu^{(3)}$ to denote respectively the second amd third raw moments of the inter-renewal times
- $B(t)$ to denote the backward recurrence time measured from some time t
- $V(t)$ to denote the forward recurrence time measured from some time t
- $W_j, j = 1, 2, \ldots$ to denote the waiting time of the j-th renewal

Note 1 A basic assumption for this work is that $\mu < \infty$. We also restrict our research to inter-renewal times that have probability density functions.

Note 2 Throughout this work the time origin is not counted as a renewal epoch. Similar works by other authors count the origin as a renewal epoch and therefore their results are not immediately comparable with the results of this work.

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By

Konstantinos Gakis

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Renewal theory has been used extensively to solve a large number of problems in science and engineering. Areas of application in Operations Research have been inventory theory, reliability theory, queueing theory, etc. B. D. Sivazlian has proposed a unified methodology based on multiple integrals to study various problems in renewal theory. Using this methodology we derive new results in the study of the backward and forward recurrence times, the probability law of the renewal counting process and the distribution of the order statistics generated by the waiting times of a renewal process. The notion of the extended compound renewal process is introduced. A Volterra type integral equation for the characteristic function of this process is used to derive the moments of this process. Several applications in infinite server queues are considered. Moreover, the concept of the extended compound point process is introduced for which expressions for the moments are derived and applications in queueing theory are investigated. Finally, some suggestions for future research are made.

CHAPTER 1

INTRODUCTION

1.1 Preview

An ordinary renewal process is a collection of independently and identically distributed random variables usually representing the time intervals between occurrences of events. A renewal counting process is a stochastic process that registers the number of events.

As the title of this dissertation –"Integral Techniques in the Study of the Ordinary Renewal Process and its Extensions"– suggests, the purpose of this work is to use known integral techniques to obtain novel results for the renewal process and its extensions, namely the extended compound renewal process and the extended compound process. In the introduction we provide the reader with basic definitions, a brief historical review of renewal theory, a brief discussion of the elementary problems in this area, the existing methodologies and with a number of applications of renewal theory in operations research. Additionally, we describe the problems of the general probability law of the renewal counting process, the order statistics of the waiting times of the ordinary renewal process and the filtered and extended compound renewal process. Finally, we provide a summary of the new results obtained and an outline of the dissertation.

1.2 The Renewal Process

According to Smith[40] "by the term *renewal process* is meant a sequence $\{X_i, i = 1, 2, \ldots\}$ of independent, nonnegative, identically distributed random variables, and to avoid triviality we suppose the X_i do not vanish with probability one. These X_i are meant to represent the lifetimes of the articles being renewed"(p. 245)(see Figure 1.1). Later the notion of renewal process was extended to allow the distribution of X_1 (or by others of X_0) to be different from that of the rest of the X_i's. To distinguish these cases, the former is called *ordinary renewal process* while the latter is called *delayed* or *modified renewal*

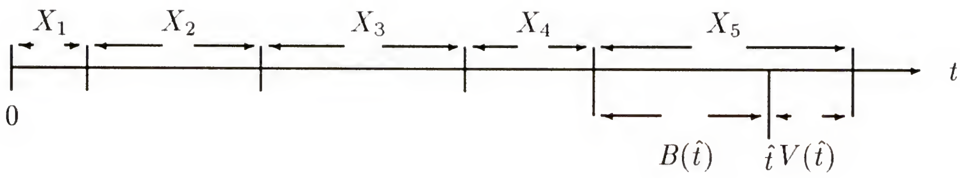

Figure 1.1. A realization of a renewal process

process. Throughout this chapter the term renewal process refers to the ordinary renewal process. As the definition suggests the renewal process can be perceived as a generalization of the Poisson process in which the sequence $\{X_i, i = 1, 2, \ldots\}$ of independent identically distributed random variables obeys the negative exponential distribution.

1.3 Renewal Theory

The origins of renewal theory –the theory concerned with the study of renewal processes– are found in the discussion of self-renewing aggregates. [1] Lotka's paper[29] –published in 1939– on *self renewing aggregates* contains a list of 74 papers on the subject of renewal equation and its applications. The oldest paper mentioned is one by Herbelot[20] who encountered the renewal equation while investigating an actuarial problem. Later Feller[11] became the first to formally study the integral equation of renewal theory. Skellam and Shenton[39] in their study of "distributions associated with random walk and recurrent events"(p. 64) obtain several results for special renewal processes. Smith[40] provides a thorough review of renewal theory. Cox[9] discusses many theoretical and applied problems in the area. Cohen[8], Daley and Vere-Jones[10] and Wolff[42] present a more modern approach to the theory.

As we already mentioned, actuarial problems have been historically the first applications of renewal theory. Population analysis and industrial replacement problems have been two other areas of application of renewal theory in its early stages. More recently, renewal theory has found applications in the areas of astronomy, astrophysics, economics, engineering, meteorology and physics.

[1] The English translation due to Lotka[29] of the German phrase "sich erneuernde Gesamtheiten," used by Swiss actuaries.

Renewal theory has found a very fruitful area of application in modeling complex systems in reliability, maintainability and availability. Renewal theory has been used for example 1) to define the operating characteristics of maintenance policies, 2) to solve the age and block replacement problem, 3) to formulate repair problems of single and multi-units, and 4) to derive optimum inspection and maintenance policies. The theory has also been applied to solve problems in reliability arising from shock processes, cumulative damages and redundancies.

Other areas of applications in operations research where renewal theory has been utilized have been single and multi-commodity inventory systems, queueing systems, maintenance and replacement systems. More recently in using diffusion approximation to solve complex queueing systems, such as the machine repair problem with standbys, renewal theory has been used to generate the infinitesimal means and variances to the diffusion equation.

1.4 Fundamentals of Renewal Theory

Related to the renewal process (as defined previously) is the *renewal counting process*[2] $\{N(t), t \geq 0\}$ which is defined to be the counting process that registers the number of renewals that occurred within a time interval $(0, t]$ (see Figure 1.2). The sum $W_k = X_1 + X_2 + \ldots + X_k$ is called the *waiting time* of the k-th renewal.

The least information one would like to know is the expectation of the number of renewals up to time t, $\mathrm{E}[N(t)]$ which is called the *renewal function* and is usually denoted by $M(t)$. Its first derivative $m(t)$ plays an important role and is known as the *renewal density*. The behavior of these two quantities has been thoroughly investigated.

Next arises the problem of determining the probability distribution of the number of renewals up to a certain time epoch t (i.e. $\Pr\{N(t) = n\}$). This has been well investigated in the literature and appears in standard textbooks.

Two other measures related to the renewal process are the backward and forward recurrence times. The *backward recurrence* time also known as the *current life*, is the time elapsed since the last renewal. The *forward recurrence* time or also known as the *excess life*,

[2]Many textbooks define the renewal process as what we call here renewal counting process.

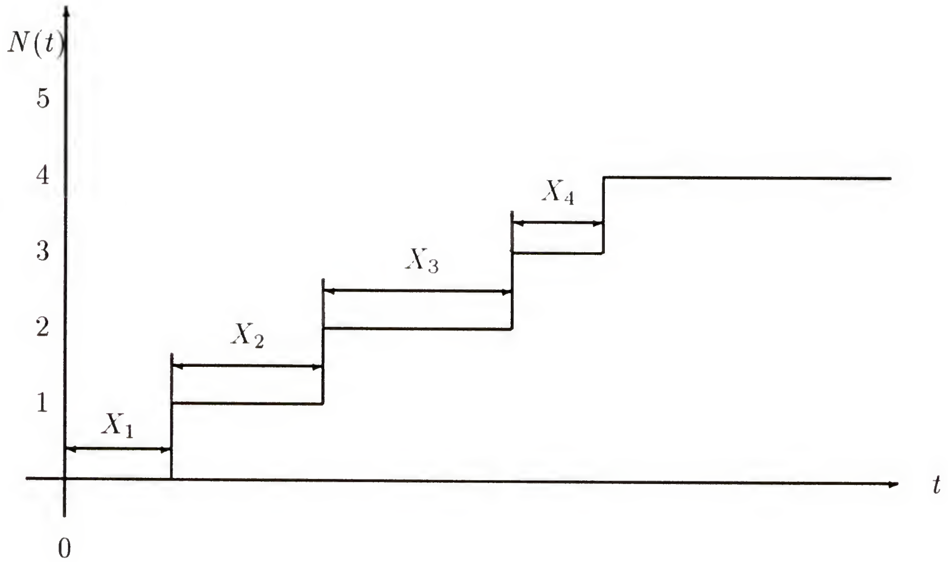

Figure 1.2. A realization of a renewal counting process

is the time that will elapse till the next renewal. The sum of the backward and forward recurrence time or the *total life* is another measure of interest.

Early research targeted the limiting behavior of the renewal process as time tends to infinity. This work obtains several results on distributions invoking the backward and forward recurrence times and their sum for the time dependent case.

A renewal counting process is fully characterized by its general probability law, that is, the joint (mass) distribution function of the number of renewals at distinct time epochs $t_1, t_2, \ldots, t_n$, where $0 < t_1 < t_2 < \ldots < t_n$ (n being an arbitrary positive integer). We discuss this in a subsequent section.

1.5 Applications of Renewal Theory

In this section we provide some applications of renewal theory, in the modeling of operations research problems. Since part of this work is concerned with backward and forward recurrence times, we motivate our discussion emphasizing applications in which these two recurrence times appear as important variables of the model.

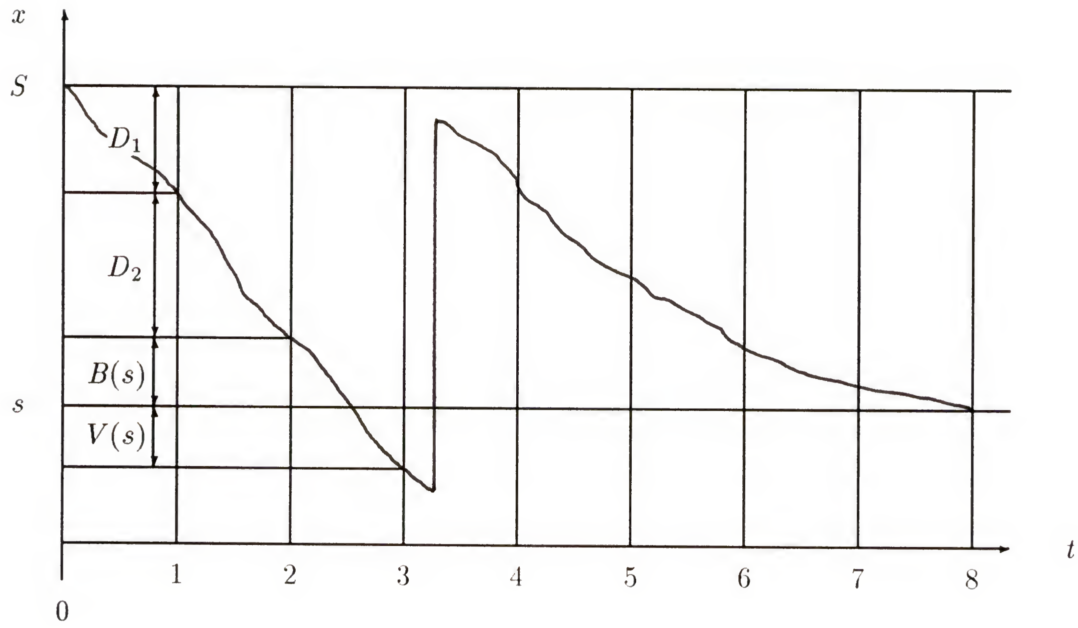

Figure 1.3. A realization of a periodic review inventory system operating under a general (s, S) policy with fixed inspection periods

1.5.1 Inventory Theory

A periodic review model. Consider a periodic review inventory system operating under a general (s, S) policy with fixed inspection periods. We assume that demands between successive periods are independent and identically distributed random variables. Moreover, let us assume instantaneous delivery of the ordered quantity. Each time the inventory level x is less than s we place an order of $(S - x)$ units that instantaneously brings the inventory level to S. Note that each time the inventory level is S, the system is in a state identical to that of any other instance in which the inventory was S. These points $(x = S)$ are called regeneration points. Let $D_i, i = 1, 2, \ldots$ denote the demands in the respective time intervals between inspection periods after a regeneration point. Clearly the D_i's form a renewal process (see Figure 1.3). If we consider the moment that the inventory hits the level s, then the inventory level at the instance of the next inspection is the forward recurrence time of the renewal process. The inventory level at the instance of the previous inspection is the backward recurrence time. Knowing the distributions of the backward and

forward recurrence times in combination with the appropriate cost functions may be used to determine the total cost function and the parameters that optimize it.

This model is a very good example of a renewal theory whose parameter does not represent time. It demonstrates the broader scope of renewal processes and the fact that they are not restricted to model systems varying over time.

A continuous review model with positive lead time. Consider a continuous review model. Demand inter-arrival times form a renewal process. The demand sizes are identically and independently distributed random variables thus forming another renewal process. The inventory replenishment policy is, whenever the inventory level falls below $s > 0$, a quantity is ordered to bring the inventory level up to a positive value $S > 0$. In general, the quantity ordered is not constant and is dictated by the demand pattern. It is always greater than or equal to $S - s$. The quantity by which it exceeds $S - s$ is the forward recurrence time of the renewal process formed by the demand at epoch $S - s$. The orders arrive after some constant time τ. Following the replenishment of an order, the time till the next demand arrives is the forward recurrence time of the renewal process formed by the inter-arrival times at epoch τ. For the use of renewal theory in modeling inventory systems the reader is referred to Arrow et al.[1]

1.5.2 Queueing Theory

A single server queue. The time intervals from the epoch at which the server becomes busy until the epoch at which he becomes idle are known as the busy cycles. Moreover, we assume that the service times are independent and identically distributed random variables denoted by $S_i, i = 1, 2, \ldots$. Clearly then these form a renewal process. One of the performance characteristics of a queueing system is the total waiting time in the queue. Let us suppose that a customer arrives during a busy cycle and finds m customers ahead of him waiting in the queue and one being presently served. Then he has to wait for the m customers to be served as well as the customer in service to complete his service. The time for m customers to be served is the sum of m independently and identically distributed random variables and the time for service completion is the forward recurrence time of the renewal process.

An infinite server queue. Consider again a queueing system with customers arriving again according to a renewal counting process. The number of servers is now, however, infinite. A customer that arrives always finds an idle server and enters for service immediately. This system can be modeled and studied by using an extension of the renewal counting process, namely the *extended compound renewal process.* We discuss this type of applications in Chapter 5.

1.5.3 Reliability Theory

Renewal theory has found some interesting applications in modeling reliability problems. We describe here a simple model utilizing renewal theory. Assume a reliability model where the component of interest suffers failures due to wear-out. We assume that the initial "strength" of this component is K_0 (where K_0 is an integer number). Each time the component suffers a wear-out its strength decreases by one. The time intervals between successive wear-outs are assumed to be independent and identically distributed random variables and thus forming a renewal process.

A frequent task of reliability models is to find the remaining life of a component at an arbitrary time epoch t_1 given that it is operating at this time. Let $T(t_1)$ denote the remaining life of the component given it has survived up to time t_1. We may know the number of wear-outs n_1 the component has suffered. Then this number is nothing more than the number of renewals at time t_1 i.e. $N(t_1) = n_1$. The time till the next wear-out is the forward recurrence time $V(t_1)$. The remaining (unused) strength is $K_r = K_0 - n_1$. Then the time to failure starting from time t_1 is

$$T = V(t_1) + X_{n_1+1} + \ldots + X_{K_0}$$

If we do not know the number of renewals up to time t_1 then the remaining strength is an integer valued random variable and the remaining life is a compound process.

For the use of renewal processes in reliability theory the reader is referred to Barlow et al.[2]

1.5.4 Group Replacement

Let us assume that we operate a large number of identical components subject to failure and we consider a group replacement policy. Each item is replaced after failure in the time interval $(0, t_1]$ (where time counts from the last group replacement). Moreover, all items are replaced at time t_1. Then the age of the components that are being replaced at time t_1 is the backward recurrence time.

1.5.5 A Maintenance Problem

We consider work to be processed on a single processor (or machine) subject to breakdown and repair. Repair is initiated as soon as the processor breaks down. At the termination of repair, the processor starts its functions immediately. Let $\{X_i,\ i = 1, 2, \ldots\}$ be the sequence of uptimes and $\{Y_i,\ i = 1, 2, \ldots\}$ the sequence of downtimes. The time to complete a piece of work of size Q is a random variable and can be described by means of an extended compound renewal process.

1.5.6 Traffic Control

In general, traffic problems can be modeled by renewal processes. We present a simple case.

Assume that the time intervals between passages of cars at an intersection are independent and identically distributed random variables. If an observer at the intersection starts observation at an arbitrary instance in time, then the time until the next car passes is the forward recurrence time. The time since the last car passed is the backward recurrence time.

1.5.7 Life Testing of Components

Consider the case when someone wants to test the life of a certain component. He may do so by operating one component until it fails and immediately replacing it by another identical component and operating it until it fails. The collection of the successive times to failure (assumed identically and independently distributed) form a renewal process. The number of failed components up to time t forms a renewal counting process. If the test stops at time t^* then the age of the component presently tested is the backward recurrence

time while its remaining life is the forward recurrence time. An interesting non-intuitive result appears in renewal theory related to life testing problems. We discuss the so-called "inspection paradox" in the next subsection.

1.5.8 The Inspection Paradox

Let us assume that life tests are performed as follows. The items are tested one at a time by a device that automatically and spontaneously replaces a failed item by a new one. Moreover, the device records the time since the last failure. An observer arrives at certain time epochs. He, immediately, records the age of the component being tested and waits until it fails. The life of the component tested is the sum of the recorded age at the moment of his arrival plus the observed time to failure, i.e. the sum of the backward and forward recurrence times. He leaves and returns several times to repeat his observation in a similar fashion. Intuitively he believes that the average of his observations can be a good estimate for the mean time to failure of the items. But, the fact is that his estimate is very biased. In fact, it will tend to be greater than the actual value of the mean time to failure. The only explanation to this is that the observer is more likely to observe larger inter-renewal times. Actually in Heyman and Sobel[21], by assuming in the limiting case that the likelihood that the observer will measure an inter-renewal time is proportional to the length of this time, one recovers the desired result. This phenomenon has been known in the literature as the *inspection paradox*. It has been investigated for the limiting case as time tends to infinity. It has also been investigated for the limiting and time dependent case for negative exponentially distributed times to failure (in which case they form a Poisson process). In this work we show that the inspection paradox exists for the time dependent case and an arbitrary distribution of inter-renewal times with probability density function and finite mean.

1.6 Methodology in the Study of Renewal Processes

Most of the past research done in the area of renewal processes has utilized the renewal equation, as a means to derive the desired results. The renewal equation, as defined by

Feller [11], is an integral equation of the form

$$A(t) = a(t) + \int_0^t A(t-u)\,dF(u)$$

or using * to denote the standard convolution operation:

$$A(t) = a(t) + A(t) * F(t)$$

Here the prescribed (or known) functions are $a(t)$ and the distribution function $F(t)$, while the undetermined (or unknown) quantity is $A(t)$. The renewal equation is readily solved, in principle, using various methods, one of which is the well known method of Laplace transforms (provided that the functions in the equation are transformable). The derivation of a renewal equation in most cases has relied on event arguments, thus restricting the scope of the methodology. B. D. Sivazlian has more recently presented a more elegant and unified methodology in approaching the problems of renewal theory (see Sivazlian[37]) using multiple integrals. This methodology is based on the following theorem:

<u>Theorem 1</u> *Define for $t > 0$, the function $g(t) \in \mathcal{C}$ (i.e. continuous) and the function $\phi_i(t) \in \mathcal{K}$ (i.e. with at most a finite number of points of discontinuity in every finite interval and such that the integral $\int_0^t |\phi_i(u)|du$ has a finite value for every $t > 0$), $i = 1, 2, \ldots, n$ where n is a positive integer. Then*

$$\int\int_{0<t_1+\cdots+t_n\le t}\cdots\int g(t_1+t_2+\cdots+t_n)\phi_1(t_1)\phi_2(t_2)\cdots\phi_n(t_n)dt_1dt_2\cdots dt_n$$

$$= \int_0^t g(u)[\phi_1(u) * \phi_2(u) * \cdots * \phi_n(u)]du$$

where the integrand of the right hand single integral is a function of class $\mathcal{K}$.

Here the notation $*$ refers to the usual convolution operation.

In this methodology we identify the region of integration (i.e. the region corresponding to the desirable event) over which we have to integrate the joint probability density function of an arbitrary number of renewals and then apply this theorem to produce our results.

The fundamental advantage of this methodology is that we can use one methodology to solve the majority of problems arising in renewal theory, and to tackle presently unsolved problems. A brief proof of the theorem in its more general form is given in the Appendix. For a complete treatment of the problem see Sivazlian[35].

1.7 The General Probability Law of the Renewal Counting Process

Let $\{X_i\}, i = 1, 2, \ldots$, be the sequence of inter-renewal times in an ordinary renewal process, assumed to be independent and identically distributed random variables with probability density function $f(x), 0 < x < \infty$, and distribution function $F(x)$. Let $\{N(t), t \geq 0\}$ be the total number of renewals in $[0, t]$ where $\Pr\{N(0) = 0\} = 1$ [3]. Consider distinct time epochs $t_1, t_2, \ldots t_m$, where $0 < t_1 < t_2 < \ldots < t_m$ and m is an arbitrary positive integer. The general probability law of the renewal counting process may be defined by

$$\Pr\{N(t_1) = n_1, N(t_2) = n_2, \ldots, N(t_m) = n_m\}$$

where $0 \leq n_1 \leq n_2 \leq \ldots \leq n_m$ for arbitrary m. Other representations of the probability law of the process, such as the joint characteristic function, may be appropriate depending on the nature of the intended results. This problem, despite its early inception and its many usages, has been considered so far to be too difficult a task to tackle and thus remained unsolved. The only special case of a renewal counting process whose general probability law has been derived is the Poisson process.

Sivazlian[36] has recently derived the joint distribution of the number of renewals at two time epochs; i.e., $\Pr\{N(t_1) = n_1, N(t_2) = n_2\}$. In the present work the joint distribution of the number of renewals at three time epochs; i.e., $\Pr\{N(t_1) = n_1, N(t_2) = n_2, N(t_3) = n_3\}$ is determined. This joint distribution function without loss of generality defines the probability law of the process.

In the case of the Poisson process the joint probability law of the renewal increments shows that they are independently distributed; i.e.,

$$\Pr\{N(t_2) - N(t_1) = n_1, N(t_3) - N(t_2) = n_2\} =$$

$$\Pr\{N(t_2) - N(t_1) = n_1\} \Pr\{N(t_3) - N(t_2) = n_2\}$$

The covariance function of the renewal increments, which is presented in this work, enables us to characterize the renewal process with respect to its increments.

[3] Several textbooks and papers assume that $\Pr\{N(0) = 1\} = 1$.

1.8 The Order Statistics of the Renewal Process

Consider a renewal process $\{X_i, i = 1, 2, \ldots\}$. Then we define the waiting times $W_i, i = 1, 2, \ldots$, of the renewals as $W_1 = X_1$, $W_2 = X_1 + X_2$, etc. Given the number of n $(n > 1)$ renewals up to time t, the order statistics is the ordered set of the waiting times $\{W_1, W_2, \ldots, W_n\}$. A problem to be investigated is the general probability law of the order statistics; i.e.,

$$\Pr\{W_1 \leq w_1, W_2 \leq w_2, \ldots, W_n \leq w_n | N(t) = n\}$$

for arbitrary n. Related problems are the distribution of the range i.e. $W_n - W_1$, the distribution of the first order statistic (the minimum), of the k-th order statistic $(1 < k < n)$, of the n-th order statistic (the maximum) and the joint probability law of their possible combinations. This problem has been investigated for the case when inter- renewal times obey the negative exponential distribution, the renewal counting process being the well-known Poisson process. For this case the probability density function of the order statistics is found to be

$$\Pr\{w_1 < W_1 \leq w_1 + dw_1, w_2 < W_2 \leq w_2 + dw_2, \ldots, w_n < W_n \leq w_n + dw_n | N(t) = n\} = \\ = \frac{n!}{t^n} dw_1\, dw_2 \ldots dw_n \tag{1.1}$$

where $0 < w_1 < w_2 < \ldots < w_n$. This is nothing more than the probability density function of the order statistics of the uniform distribution.

This result finds application in the treatment of many problems including the study of the filtered Poisson process(see Parzen[31]) and networks of infinite server queues (see Harrison and Lemoine[19]). The distribution of the order statistics may be helpful in the study of similar problems in renewal theory. The expression corresponding to (1.1) when the process is an ordinary renewal counting process will be obtained in the sequel.

1.9 Extensions of the Renewal Process

We first define the filtered Poisson process:

A stochastic process $\{X(t), t \geq 0\}$ is said to be a *filtered Poisson process* if it can be represented by

$$\begin{aligned} X(t) &= \sum_{m=1}^{N(t)} w(t, W_m, Y_m) \\ &= w(t, W_1, Y_1) + w(t, W_2, Y_2) + \ldots w(t, W_{N(t)}, Y_{N(t)}) \end{aligned} \tag{1.2}$$

where

1. $\{N(t), t \geq 0\}$ is a Poisson process with intensity λ;

2. $\{Y_m,\ m = 1, 2, \ldots\}$ is a sequence of independently and identically distributed random variables, independent of $\{N(t), t \geq 0\}$;

3. $\{W_m,\ m = 1, 2, \ldots\}$ is a sequence of waiting times of the Poisson process $\{N(t), t \geq 0\}$;

4. $w(t, x, y)$ is a real valued function of three variables and is called the response function.

Parzen[31] gives the following physical interpretation for the filtered Poisson process:

> if W_m represents the time at which an event took place, then Y_m represents the amplitude of a signal associated with the event, $w(t, W_m, Y_m)$ represents the value at time t of a signal of magnitude Y_m originating at time W_m, and $X(t)$ represents the value at time t of the sum of the signals arising from the events occurring in the interval $(0, t\,]$.

For more information to the theory of filtered Poisson processes the reader is referred to Blanc-Lapierre and Fortet[3], Parzen[31] and Karlin and Taylor[25].

A natural extension of the concept of the filtered Poisson process is the filtered renewal process. The *filtered renewal process* is defined similarly to the filtered Poisson process, only now $\{N(t), t \geq 0\}$ is a renewal counting process.

A stochastic process related to the filtered renewal process is the cumulative process defined as follows.

Let $\{(X_k, Y_k), k = 1, 2, \ldots\}$ be a sequence of independent and identically distributed bivariate random variables with joint distribution function $\phi(x, y)$. Suppose, moreover, that the X_k's are strictly positive thus forming a renewal process and that $\{N(t), t > 0\}$ is the renewal counting process related to them. Then

$$Z(t) = \sum_{j=1}^{N(t)+1} Y_j$$

is a *cumulative process.*

Clearly, if in addition we assume that the X_k's are independently distributed of the Y_k's, the cumulative process is a special case of the filtered renewal process with response function

$$w(t, W_k, Y_k) = Y_k$$

Cumulative processes have found many applications in operations research problems such as inventory and reliability models. Recently, Roginsky[34] has presented some interesting asymptotic results on renewal and cumulative processes.

The notion of the filtered Poisson process can be extended if we let

$$\begin{aligned} X(t) &= \sum_{m=1}^{N(t)} Y_m(t, W_m) \\ &= Y_1(t, W_1) + Y_2(t, W_2) + \ldots Y_{N(t)}(t, W_{N(t)}) \end{aligned} \tag{1.3}$$

where $\{Y_m(t, W_m), t \geq W_m\}$ is now a sequence of *stochastic processes* (see Parzen[31]). We call similar process where the underlying point process is a renewal process, an *extended compound renewal process.*

The primary reason for being unable to extend the theory of the filtered Poisson process has been the unavailability of mathematical techniques to handle the complex multiple integral expressions that arise in related problems. For an extensive introductory discussion on the filtered renewal process, the reader is referred to Sivazlian[37].

1.10 Objectives and New Results

We make use of a unifying methodology to solve many complex time dependent problems in renewal theory whose solutions have eluded researchers so far. This methodology is based on the result in multiple integrals established by Sivazlian[35] which was presented in Section 1.6. Multiple integrals provide a natural vehicle to approach these complex problems as one is essentially dealing with sums of independent random variables in the context of inter-renewal times.

This methodology can be used as a new way for deriving most existing results for the time dependent renewal counting process. Our objective is to show that it can also be used to provide solutions to many unsolved problems, thus bringing forth a new perspective to the

analysis of renewal processes. Existing methodologies have only resulted in a partial characterization of the properties of renewal processes since they invariably define the marginal distribution of a particular random variable. The new methodology characterizes more fully these processes by finding joint distribution functions. The analysis of a process is incomplete unless one has studied and derived joint distribution functions such as the probability law of the process. In the case of renewal processes, a partial characterization of the probability law ($\Pr\{N(t_1) = n_1, N(t_2) = n_2\}$) was established by Sivazlian[36]. Our objective will be to completely characterize this law ($\Pr\{N(t_1) = n_1, N(t_2) = n_2, N(t_3) = n_3\}$). It is to be noted that such characterizations already exist for the case of the Poisson process with negative exponentially distributed inter-renewal times. We extend these results to general inter-renewal times and we derive the joint distribution of the renewal increments and investigate their asymptotic properties.

The additional new results that are derived in the area of backward and forward recurrence times fill in important existing gaps in the study of renewal processes, particularly in obtaining the joint distribution function of the backward and forward recurrence times and the number of renewals. These results are used to generalize the inspection paradox in the very general case of the time dependent problem and arbitrary distribution of inter-renewal times.

We derive important results such as the distribution of order statistics for the waiting times in an ordinary renewal process.

Another problem area which has received no attention is that of the extended compound renewal process, a generalization of the concept of filtered Poisson process in the sense of Parzen[31]. A second type Volterra integral equation is obtained for the characteristic function of this process. From this equation closed form and quasi-closed form expressions are obtained for the moments of this process and several of its many applications are discussed with particular emphasis to infinite server queues.

The notion of the extended compound renewal process is further generalized to that of the extended compound process for which we again obtain quasi-closed form expressions for its moments. We provide an example that demonstrates the usefulness of our approach

to model systems that could not have been modeled using previously available analytical models.

Integral techniques used. The unifying theme of this dissertation is the use of integral techniques as a common tool to derive new results in renewal theory. The integral techniques used can be classified into three basic categories: multiple integrals, integral equations and generalization of the Stieltjes integral to several variables. We briefly ellaborate on each of these techniques.

In Chapters 2, 3 and 4 we use the multiple integral technique based on Theorem 1 of this chapter. We have elaborated on the importance of manipulation of multiple integrals in the study of ordinary renewal processes. In Chapter 5 we use a second type Volterra integral equation in order to investigate the characteristic function and the moments of the extended compound renewal process. We remind the reader that the renewal equation is a special case of Volterra equation of the second type. Additionally, the notion and the properties of the multiple Stieltjes integral is used to derive the moments of the more general extended compound point process.

The results obtained exemplify the type of theoretical knoweledge acquired by this research and how such knowledge can be used to understand more fully the behavior of renewal processes and its ramifications. In particular asymptotic properties can be derived simply and directly from the many time dependent solutions obtained, thus obviating complex derivations used so far and which have mostly relied on the use of mathematical analysis.

1.11 Outline

In Chapter 1, the origins of renewal theory are reviewed and definitions of the renewal process and the renewal counting process are provided. Fundamental problems as well as existing and new methodologies for solving them are discussed. Applications of renewal theory and novel problems and extensions are presented.

In Chapter 2, the joint distributions of backward and forward recurrence times are investigated. Their covariance function is determined. The existence of the inspection paradox for the time dependent case is demonstrated.

The joint and marginal distributions of the order statistics of waiting times of an ordinary renewal process are studied in Chapter 3.

In Chapter 4, the joint distribution of the number of renewals at three time epochs is derived. Hence the joint distribution of the renewal increments is determined and asymptotic properties are investigated. The covariance function of the renewal increments is also determined.

The filtered renewal process and the extended compound renewal process are defined in Chapter 5. The characterisitc function of a form of the extended compound renewal process is shown to be the solution of a second type Volterra integral equation. Its solution is determined for some special cases. The expectation and recursive relations for higher order moments are determined. Applications to infinite server queues are provided. The notion of the extended compound renewal process is generalized to the extended compound point process. The moments for that general process are determined.

Finally, in Chapter 6, suggestions for future research are made. Specifically, potential applications of the notion of the extended compound renewal process in inventory theory for an (s, S) policy model and in queueing theory for tandem infinite server queues are discussed. The problems of first passage times of the compound renewal process and of the joint characteristic process of the extended compound point process are introduced. Additionally, the problem of determining a *parent distribution* of the order statistics of the waiting times of the renewal process is introduced. Such a parent distribution when sampled will have the order statistics of the waiting times of the renewal process.

CHAPTER 2

THE USE OF MULTIPLE INTEGRALS IN THE STUDY OF THE BACKWARD AND FORWARD RECURRENCE TIMES

2.1 Introduction

A pioneer work in the study of the backward and forward recurrence times and their sum is that of Skellam and Shenton[39]. The marginal distribution functions of the backward and forward recurrence times and their sum for both the time dependent and steady state case are known in the literature. The joint distribution of the backward and forward recurrence times for the stationary renewal process has also been investigated (see e.g. Cohen[8] and Daley and Vere-Jones[10]). Daley and Vere-Jones give a result for the time dependent case that is not clearly stated. The joint distribution of the number of renewals and the backward and forward recurrence times does not seem to have been investigated so far. In a previous work, Sivazlian[38] shows how multiple integrals can be used to derive the joint distribution functions of the number of renewals and the backward and of the forward recurrence times separately. In this paper, we use again a result in multiple integrals introduced by Sivazlian[35] to derive the *time dependent joint distribution function of the number of renewals and the backward and forward recurrence times* for an ordinary renewal process. Thus we recover the joint distribution function and the marginal distribution functions of the backward and forward recurrence times. The *covariance* of the backward and forward recurrence times is derived from their complementary joint distribution function for both the time dependent and the steady state cases. We also obtain the *time dependent joint distribution function of the number of renewals and the sum of the backward and forward recurrence times.* The expectation of the sum of the backward and forward recurrence times is derived and the *inspection paradox for the time dependent case* is investigated.

For the use of multiple integrals in renewal theory the reader is referred to Sivazlian[36, 37, 38]

2.2 The Joint Distribution Function of the Number of Renewals and the Backward and Forward Recurrence Times

In order to find the joint distribution of the backward and forward recurrence times and the number of renewals, i.e. $\Pr\{N(t) = n, B(t) \leq \theta, V(t) \leq \tau\}$ we consider two cases.

Case 1. $N(t) = 0$. Clearly then $B(t) = t$ and therefore

1. For $0 \leq \theta < t$

$$\Pr\{N(t) = 0, B(t) \leq \theta, V(t) \leq \tau\} = 0 \tag{2.1}$$

2. For $t \leq \theta < \infty$

$$\begin{aligned} \Pr\{N(t) = 0, B(t) \leq \theta, V(t) \leq \tau\} &= \Pr\{N(t) = 0, V(t) \leq \tau\} \\ &= F(t+\tau) - F(t) \end{aligned} \tag{2.2}$$

Case 2. $N(t) = n$ where $n = 1, 2, \ldots$. Then $B(t) = t - (T_1 + T_2 + \cdots + T_n)$

1. For $0 \leq \theta < t$

$$\begin{aligned} &\Pr\{N(t) = n, B(t) \leq \theta, V(t) \leq \tau\} \\ &= \Pr\{N(t) = n, T_1 + T_2 + \cdots + T_{n+1} \leq t + \tau, T_1 + T_2 + \cdots + T_n > t - \theta\} \\ &= \Pr\{T_1 + T_2 + \cdots + T_n \leq t, T_1 + T_2 + \cdots + T_{n+1} > t, \\ &\qquad T_1 + T_2 + \cdots + T_{n+1} \leq t + \tau, T_1 + T_2 + \cdots + T_n > t - \theta\} \\ &= \underset{t-\theta < t_1 + \cdots + t_n \leq t}{\int\int\cdots\int} f(t_1)f(t_2)\cdots f(t_n) \\ &\qquad \{F[t+\tau-(t_1+t_2+\cdots+t_n)] - F[t-(t_1+t_2+\cdots+t_n)]\}\, dt_1\, dt_2 \cdots dt_n \\ &= \int_{t-\theta}^{t} f^{*(n)}(u)[F(t+\tau-u) - F(t-u)]du \end{aligned} \tag{2.3}$$

2. For $t \leq \theta < \infty$

$$\begin{aligned} &\Pr\{N(t) = n, B(t) \leq \theta, V(t) \leq \tau\} \\ &= \Pr\{N(t) = n, V(t) \leq \tau\} \\ &= \Pr\{N(t) = n, T_1 + T_2 + \cdots + T_{n+1} \leq t + \tau, \} \\ &= \Pr\{T_1 + T_1 + \cdots + T_n \leq t, T_1 + T_2 + \cdots + T_{n+1} > t, \\ &\qquad T_1 + T_2 + \cdots + T_{n+1} \leq t + \tau, \} \end{aligned}$$

$$= \int\int_{0<t_1+\cdots+t_n\le t}\cdots\int f(t_1)f(t_2)\cdots f(t_n)$$
$$\{F[t+\tau-(t_1+\cdots+t_n)]-F[t-(t_1+\cdots+t_n)]\}\,dt_1\,dt_2\cdots dt_n$$
$$= \int_0^t f^{*(n)}(u)[F(t+\tau-u)-F(t-u)]du \tag{2.4}$$

Therefore summarizing we have

$$\Pr\{N(t)=0, B(t)\le\theta, V(t)\le\tau\} = \begin{cases} 0, & 0\le\theta<t \\ F(t+\tau)-F(t), & t\le\theta<\infty \end{cases} \tag{2.5}$$

And for $n = 1, 2, \ldots$

$$\Pr\{N(t)=n, B(t)\le\theta, V(t)\le\tau\} =$$
$$\begin{cases} \int_{t-\theta}^{t} f^{*(n)}(u)[F(t+\tau-u)-F(t-u)]du, & 0\le\theta<t \\ \int_0^t f^{*(n)}(u)[F(t+\tau-u)-F(t-u)]du, & t\le\theta<\infty \end{cases} \tag{2.6}$$

2.3 The Joint Distribution Function of the Backward and Forward Recurrence Times

2.3.1 The Time Dependent Case

Combining equations (2.5) and (2.6), we have for $0\le\tau<\infty$

$$\Pr\{B(t)\le\theta, V(t)\le\tau\} = \sum_{n=0}^{\infty}\Pr\{N(t)=n, B(t)\le\theta, V(t)\le\tau\}$$
$$= \begin{cases} \sum_{n=1}^{\infty}\int_{t-\theta}^{t} f^{*(n)}(u)[F(t+\tau-u)-F(t-u)]du, & 0\le\theta<t \\ F(t+\tau)-F(t)+\sum_{n=1}^{\infty}\int_0^t f^{*(n)}(u)[F(t+\tau-u)-F(t-u)]du, & t\le\theta<\infty \end{cases}$$

Since

$$m(t) = \sum_{n=1}^{\infty} f^{*(n)}(t)$$

then for $0\le\tau<\infty$

$$\Pr\{B(t)\le\theta, V(t)\le\tau\}$$
$$= \begin{cases} \int_{t-\theta}^{t} m(u)[F(t+\tau-u)-F(t-u)]du, & 0\le\theta<t \\ F(t+\tau)-F(t)+\int_0^t m(u)[F(t+\tau-u)-F(t-u)]du, & t\le\theta<\infty \end{cases}$$

$$= \begin{cases} \int_0^\theta m(t-v)[F(\tau+v)-F(v)]dv, & 0 \le \theta < t \\ F(t+\tau)-F(t)+\int_0^t m(t-v)[F(\tau+v)-F(v)]dv, & t \le \theta < \infty \end{cases} \tag{2.7}$$

We can elaborate more on the second part of this equation:

$$\begin{aligned} & F(t+\tau)-F(t)+\int_0^t m(t-v)[F(\tau+v)-F(v)]dv = \\ & = F(t+\tau)-F(t)+\int_0^t m(t-v)F(\tau+v)dv - \int_0^t m(t-v)F(v)dv \\ & = F(t+\tau)-F(t)+\int_0^t m(t-v)F(\tau+v)dv - M(t)+F(t) \\ & = F(t+\tau)+\int_0^t m(t-v)F(\tau+v)dv - \int_0^t m(t-v)dv \\ & = F(t+\tau)-\int_0^t [1-F(\tau+v)]m(t-v)dv \end{aligned}$$

Thus, we can write,

$$\Pr\{B(t) \le \theta, V(t) \le \tau\} = \begin{cases} \int_0^\theta m(t-v)[F(\tau+v)-F(v)]dv, & 0 \le \theta < t \\ F(t+\tau)-\int_0^t m(t-v)[1-F(\tau+v)]dv, & t \le \theta < \infty \end{cases} \tag{2.8}$$

2.3.2 The Limit as $t \to \infty$

It immediately follows that

$$\begin{aligned} \lim_{t\to\infty} \Pr\{B(t) \le \theta, V(t) \le \tau\} &= \lim_{t\to\infty} \int_0^\theta m(t-v)[F(\tau+v)-F(v)]dv \\ &= \frac{1}{\mu}\int_0^\theta [F(\tau+v)-F(v)]dv \end{aligned} \tag{2.9}$$

where $0 \le \theta < \infty$ and $0 \le \tau < \infty$. A similar result is presented by Cohen[8].

2.4 The Marginal Distribution Functions of the Backward and Forward Recurrence Times

The distribution functions of the backward recurrence time and of the forward recurrence time are results that can be found in the literature(see e.g. Sivazlian[38]). Here, we derive them as the marginals of the joint distribution function of the backward and forward recurrence times.

2.4.1 The Marginal Distribution Function of the Backward Recurrence Time

The marginal distribution function of the backward recurrence time can be found if we let $\tau \to \infty$ in equation (2.8).

$$\Pr\{B(t) \le \theta\} = \lim_{\tau\to\infty} \Pr\{B(t) \le \theta, V(t) \le \tau\}$$

$$= \begin{cases} \lim_{\tau\to\infty} \int_0^\theta m(t-v)[F(\tau+v)-F(v)]dv, & 0 \le \theta < t \\ \lim_{\tau\to\infty} \{F(t+\tau) - \int_0^t m(t-v)[1-F(\tau+v)]dv\}, & t \le \theta < \infty \end{cases}$$

$$= \begin{cases} \int_0^\theta m(t-v)[1-F(v)]dv, & 0 \le \theta < t \\ 1, & t \le \theta < \infty \end{cases} \tag{2.10}$$

2.4.2 The Marginal Distribution Function of the Forward Recurrence Time

The marginal distribution of the forward recurrence time can be found if we let $\theta \to \infty$ in equation (2.5).

$$\begin{aligned} \Pr\{V(t) \le \tau\} &= \lim_{\theta\to\infty} \Pr\{B(t) \le \theta, V(t) \le \tau\} \\ &= F(t+\tau) - \int_0^t m(t-v)[1-F(\tau+v)]dv \end{aligned} \tag{2.11}$$

2.5 The Complementary Joint Distribution Function of the Number of Renewals and the Backward and Forward Recurrence Times

The complementary joint distribution function of the number of renewals and the backward and forward recurrence times, i.e. $\Pr\{N(t) = n, B(t) > \theta, V(t) > \tau\}$, can be derived from equation (2.5). Here we present an alternative derivation using the joint distribution function of the number of renewals as it is presented by Sivazlian[36].

Again we consider two cases.

Case 1. $N(t) = 0$. Clearly then for $0 \le \tau < \infty$

1. For $0 \le \theta < \infty$

$$\begin{aligned} \Pr\{N(t) = 0, B(t) > \theta, V(t) > \tau\} &= \Pr\{N(t) = 0, N(t+\tau) = 0\} \\ &= \Pr\{T_1 > t+\tau\} \\ &= 1 - F(t+\tau) \end{aligned} \tag{2.12}$$

2. For $t \le \theta < \infty$

$$\Pr\{N(t) = 0, B(t) > \theta, V(t) > \tau\} = 0 \tag{2.13}$$

Case 2. $N(t) = n$ where $n = 1, 2, \ldots$. Then for $0 \le \tau < \infty$

1. For $0 \le \theta < t$

$$\Pr\{N(t) = n, B(t) > \theta, V(t) > \tau\} =$$

$$= \Pr\{N(t-\theta) = n, N(t+\tau) = n\}$$
$$= \int_0^{t-\theta} f^{*(n)}(u)[1 - F(t+\tau-u)]du \tag{2.14}$$

2. For $t \leq \theta < \infty$

$$\Pr\{N(t) = n, B(t) > \theta, V(t) > \tau\} = 0 \tag{2.15}$$

Thus summarizing we have

$$\Pr\{N(t) = 0, B(t) > \theta, V(t) > \tau\} = \begin{cases} 1 - F(t+\tau), & 0 \leq \theta < t \\ 0, & t \leq \theta < \infty \end{cases} \tag{2.16}$$

And for $n = 1, 2, \ldots$

$$\Pr\{N(t) = n, B(t) > \theta, V(t) > \tau\} = \begin{cases} \int_0^{t-\theta} f^{*(n)}(u)[1 - F(t+\tau-u)]du, & 0 \leq \theta < t \\ 0, & t \leq \theta < \infty \end{cases} \tag{2.17}$$

2.6 The Complementary Joint Distribution Function of the Backward and Forward Recurrence Times

2.6.1 The Time Dependent Case

Combining equations (2.16) and (2.17) we get for $0 \leq \tau < \infty$

$$\Pr\{B(t) > \theta, V(t) > \tau\} = \sum_{n=0}^{\infty} \Pr\{N(t) = n, B(t) > \theta, V(t) > \tau\}$$
$$= \begin{cases} 1 - F(t+\tau) + \sum_{n=0}^{\infty} \int_0^{t-\theta} f^{*(n)}[1 - F(t+\tau-u)]du, & 0 \leq \theta < t \\ 0, & t \leq \theta < \infty \end{cases}$$
$$= \begin{cases} 1 - F(t+\tau) + \int_0^{t-\theta} m(u)[1 - F(t+\tau-u)]du, & 0 \leq \theta < t \\ 0, & t \leq \theta < \infty \end{cases}$$
$$= \begin{cases} 1 - F(t+\tau) + \int_\theta^{t} m(t-v)[1 - F(v+\tau)]dv, & 0 \leq \theta < t \\ 0, & t \leq \theta < \infty \end{cases} \tag{2.18}$$

The marginal complementary distribution functions can be obtained if we let $\tau = 0$ and $\theta = 0$ accordingly.

$$\Pr\{B(t) > \theta\} = \begin{cases} 1 - F(t) + \int_\theta^{t} m(t-v)[1 - F(v)]dv, & 0 \leq \theta < t \\ 0, & t \leq \theta < \infty \end{cases} \tag{2.19}$$

And

$$\Pr\{V(t) > \tau\} = 1 - F(t+\tau) + \int_0^{t} m(t-v)[1 - F(v+\tau)]dv, \; 0 \leq \tau < \infty \tag{2.20}$$

2.6.2 The Limit as $t \to \infty$

If we let $t \to \infty$ in equation (2.18) we have

$$
\begin{aligned}
&\lim_{t\to\infty} \Pr\{B(t) > \theta, V(t) > \tau\} = \\
&= \lim_{t\to\infty} \{1 - F(t+\tau) + \int_\theta^t m(t-v)[1 - F(v+\tau)]dv\} \\
&= \lim_{t\to\infty} \{1 - F(t+\tau) + \int_0^t m(t-v)[1 - F(v+\tau)]dv\} \\
&\qquad\qquad - \int_0^\theta m(t-v)[1 - F(v+\tau)]dv \qquad (2.21)
\end{aligned}
$$

Using Smith's Theorem this last equation becomes

$$
\begin{aligned}
&\lim_{t\to\infty} \Pr\{B(t) > \theta, V(t) > \tau\} = \\
&= \frac{1}{\mu}\int_0^\infty [1 - F(v+\tau)]dv - \frac{1}{\mu}\int_0^\theta [1 - F(v+\tau)]dv \\
&= \frac{1}{\mu}\int_\theta^\infty [1 - F(v+\tau)]dv \\
&= \frac{1}{\mu}\int_{\theta+\tau}^\infty [1 - F(v)]dv \qquad (2.22)
\end{aligned}
$$

The limits of the marginal complementary distribution functions follow immediately if we let $\tau = 0$ and $\theta = 0$ accordingly.

$$\lim_{t\to\infty} \Pr\{B(t) > \theta\} = \frac{1}{\mu}\int_\theta^\infty [1 - F(v)]dv \qquad (2.23)$$

And

$$\lim_{t\to\infty} \Pr\{V(t) > \tau\} = \frac{1}{\mu}\int_\tau^\infty [1 - F(v)]dv \qquad (2.24)$$

<u>Example 1</u> *The Poisson process*

As an example we investigate the case when the inter-renewal times obey the negative exponential distribution with parameter λ and consequently mean inter-renewal time $\mu = 1/\lambda$. For this negative exponential distribution,

$$F(t) = 1 - e^{-\lambda t}$$

and

$$m(t) = \lambda$$

So substituting into equations (2.5) and (2.6) we obtain

$$\Pr\{N(t)=0, B(t)\le\theta, V(t)\le\tau\} = \begin{cases} 0, & 0\le\theta<\infty \\ e^{-\lambda t}-e^{-\lambda(t+\tau)}, & t\le\theta<\infty \end{cases}$$

And for $n=1,2,\ldots$ and $0\le\theta<t, 0\le\tau<\infty$

$$\begin{aligned}
&\Pr\{N(t)=n, B(t)\le\theta, V(t)\le\tau\} = \\
&= \int_{t-\theta}^{t} \frac{\lambda^n u^{n-1}e^{-\lambda u}}{(n-1)!}[1-e^{-\lambda(t+\tau-u)}-1+e^{-\lambda(t-u)}]du \\
&= \int_{t-\theta}^{t} \frac{\lambda^n u^{n-1}e^{-\lambda u}}{(n-1)!}e^{-\lambda(t-u)}[1-e^{-\lambda\tau}]du \\
&= e^{-\lambda t}(1-e^{-\lambda\tau})\int_{t-\theta}^{t}\frac{\lambda^n u^{n-1}}{(n-1)!}du \\
&= (1-e^{-\lambda\tau})\frac{\lambda^n[t^n-(t-\theta)^n]e^{-\lambda t}}{n!}
\end{aligned}$$

Similarly for $t\le\theta<\infty, 0\le\tau<\infty$

$$\Pr\{N(t)=n, B(t)\le\theta, V(t)\le\tau\} = (1-e^{-\lambda t})\frac{\lambda^n t^n e^{-\lambda t}}{n!}$$

Substituting in equations (2.8) and (2.9) we obtain

$$\Pr\{V(t)\le\theta, V(t)\le\tau\} = \begin{cases} \displaystyle\int_0^\theta \lambda[e^{-\lambda v}-e^{-\lambda(\tau+v)}]dv, & 0\le\theta<\infty \\ \displaystyle 1-e^{-\lambda(t+\tau)}-\int_0^t \lambda[1-1+e^{-\lambda(\tau+v)}]dv, & t\le\theta<\infty \end{cases}$$

$$= \begin{cases} \displaystyle(1-e^{-\lambda\tau})\int_0^\theta \lambda e^{-\lambda v}dv, & 0\le\theta<\infty \\ \displaystyle 1-e^{-\lambda(t+\tau)}-\int_0^t \lambda e^{-(\tau+v)}dv, & t\le\theta<\infty \end{cases}$$

$$= \begin{cases} (1-e^{-\lambda\theta})(1-e^{-\lambda\tau}), & 0\le\theta<t \\ 1-e^{-\lambda t}, & t\le\theta<\infty \end{cases}$$

As we easily see the joint distribution function of the backward and forward recurrence times is dependent on the time parameter t. Nevertheless the backward and forward recurrence times are independently distributed.

Similarly substituting in equation (2.18) we can see that

$$\Pr\{B(t)>\theta, V(t)>\tau\} =$$

$$\begin{cases} \displaystyle 1-1+e^{-\lambda(t+\tau)}+\int_\theta^t \lambda[1-1+e^{-\lambda(v+\tau)}]dv, & 0\le\theta<t \\ 0, & t\le\theta<\infty \end{cases}$$

$$= \begin{cases} e^{-\lambda(t+\tau)} + \int_\theta^t \lambda e^{-\lambda(v+\tau)} dv, & 0 \le \theta < t \\ 0, & t \le \theta < \infty \end{cases}$$

$$= \begin{cases} e^{-\lambda(t+\tau)} + e^{-\lambda(\theta+\tau)} - e^{-\lambda(t+\tau)}, & 0 \le \theta < t \\ 0, & t \le \theta < \infty \end{cases}$$

$$= \begin{cases} e^{-\lambda(\theta+\tau)}, & 0 \le \theta < t \\ 0, & t \le \theta < \infty \end{cases}$$

2.7 The Covariance of $B(t)$ and $V(t)$

The covariance of the backward and forward recurrence times does not seem to have appeared so far. Here we use the joint and the marginal complementary distribution functions of the backward and forward recurrence times to derive expressions for both the time dependent and the steady state cases.

2.7.1 The Time Dependent Case

First we find an expression for $\mathrm{E}[B(t)V(t)]$:

$$\mathrm{E}[B(t)V(t)] = \int_0^\infty \int_0^\infty \Pr\{B(t) > x, V(t) > y\} dy dx$$

$$= \int_0^t \int_0^\infty [1 - F(t+y)] dy dx + \int_0^t \int_0^\infty \int_0^{t-x} [1 - F(t+y-u)] du dy dx$$

$$= \int_0^t \int_t^\infty [1 - F(v)] dv dx + \int_0^t \int_0^{t-x} \int_{t-u}^\infty [1 - F(v)] m(u) dv du dx \tag{2.25}$$

If we let

$$P(t) = \int_t^\infty [1 - F(v)] dv$$

the last equation becomes

$$\mathrm{E}[B(t)V(t)] = \int_0^t P(t) dx + \int_0^t \int_0^{t-x} P(t-u) m(u) du dx$$

$$= tP(t) + \int_0^t \int_0^{t-u} P(t-u) m(u) dx du$$

$$= tP(t) + \int_0^t (t-u) P(t-u) m(u) du$$

$$= tP(t) + [tP(t)] * m(t) \tag{2.26}$$

Next we find expresions for $\mathrm{E}[B(t)]$ and $\mathrm{E}[V(t)]$:

$$\mathrm{E}[B(t)] = \int_0^\infty \Pr\{B(t) > x\} dx =$$

$$
\begin{aligned}
&= \int_0^t [1 - F(t)]dx + \int_0^t \int_x^t m(t-v)[1-F(v)]dvdx \\
&= t[1 - F(t)] + \int_0^t \int_0^v m(t-v)[1-F(v)]dxdv \\
&= t[1 - F(t)] + \int_0^t v[1-F(v)]m(t-v)dv \qquad (2.27)
\end{aligned}
$$

If we let

$$R(t) = 1 - F(t)$$

the last equation becomes

$$E[B(t)] = tR(t) + [tR(t)] * m(t) \qquad (2.28)$$

Similarly,

$$
\begin{aligned}
E[V(t)] &= \int_0^\infty \Pr\{V(t) > y\} \\
&= \int_0^\infty [1 - F(t+y)]dy + \int_0^\infty \int_0^t m(t-v)[1-F(t+y)]dvdy \\
&= P(t) + P(t) * m(t) \qquad (2.29)
\end{aligned}
$$

Combining relations (2.26), (2.28) and (2.29) we have

$$\mathrm{Cov}[B(t), V(t)] = E[B(t)V(t)] - E[B(t)]E[V(t)]$$

$$= tP(t) + [tP(t)] * m(t) - \{P(t) + P(t) * m(t)\}\{tR(t) + [tR(t)] * m(t)\} \qquad (2.30)$$

Note that although this is not a closed form expression, nevertheless the quantities in the right-hand side of equation (2.30) have a structure similar to that of the renewal equation and thus we can easily obtain their Laplace transforms.

<u>Example 2</u> *The Poisson process*

Again as an example we calculate the covariance function for the negative exponential distribution.

$$P(t) = \int_t^\infty [1 - 1 + e^{-\lambda v}]dv = \frac{e^{-\lambda t}}{\lambda}$$

And

$$
\begin{aligned}
[tP(t)] * m(t) &= \int_0^t u\frac{e^{-\lambda u}}{\lambda}\lambda du = \int_0^t ue^{-\lambda u}du \\
&= -\frac{e^{-\lambda t}}{\lambda}(t + \frac{1}{\lambda}) + \frac{1}{\lambda^2}
\end{aligned}
$$

So,

$$\mathrm{E}[B(t)V(t)] = \frac{te^{-\lambda t}}{\lambda} - \frac{e^{-\lambda t}}{\lambda}(t + \frac{1}{\lambda}) + \frac{1}{\lambda^2} = \frac{1}{\lambda^2}(1 - e^{-\lambda t})$$

Also,

$$\mathrm{E}[B(t)] = te^{-\lambda t} + \int_0^t \lambda te^{-\lambda t}dt = te^{-\lambda t}(t + \frac{1}{\lambda}) + \frac{1}{\lambda} = \frac{1}{\lambda}(1 - e^{-\lambda t})$$

And

$$\mathrm{E}[V(t)] = \frac{e^{-\lambda t}}{\lambda} + \lambda \int_0^t \frac{e^{-\lambda t}}{\lambda}dt = \frac{e^{-\lambda t}}{\lambda} + \frac{1 - e^{-\lambda t}}{\lambda} = \frac{1}{\lambda}$$

Thus we conclude that

$$\mathrm{Cov}[B(t), V(t)] = \frac{1}{\lambda^2}(1 - e^{-\lambda t}) - \frac{1}{\lambda^2}(1 - e^{-\lambda t}) = 0$$

which is an immediate consequence of the independence of the backward and forward recurrence times.

2.7.2 The Steady State Case

Here we use the limits as $t \to \infty$ of the complementary joint distribution function of the backward and forward recurrence times as given by equation (2.18)

$$\begin{aligned}
\lim_{t\to\infty} \mathrm{E}[B(t)V(t)] &= \frac{1}{\mu}\int_0^\infty \int_0^\infty \int_\theta^\infty [1 - F(v + \tau)]dvd\theta d\tau \\
&= \frac{1}{\mu}\int_0^\infty \int_0^\infty \int_0^v [1 - F(v + \tau)]d\theta dvd\tau \\
&= \frac{1}{\mu}\int_0^\infty \int_0^\infty v[1 - F(v + \tau)]dvd\tau \\
&= \frac{1}{\mu}\int_0^\infty \int_v^\infty v[1 - F(x)]dxdv \\
&= \frac{1}{\mu}\int_0^\infty \int_0^x v[1 - F(x)]dvdx \\
&= \frac{1}{\mu}\int_0^\infty \frac{x^2}{2}[1 - F(x)]dx \\
&= \frac{\mu^{(3)}}{6\mu} \qquad (2.31)
\end{aligned}$$

Moreover we know that

$$\lim_{t\to\infty} \mathrm{E}[B(t)] = \lim_{t\to\infty} \mathrm{E}[V(t)] = \frac{\mu^{(2)}}{2\mu}$$

Therefore we can write

$$\lim_{t\to\infty} \mathrm{Cov}[B(t)V(t)] = \frac{\mu^{(3)}}{6\mu} - \frac{(\mu^{(2)})^2}{4\mu^2} \qquad (2.32)$$

Example 3 *Gamma distributed inter-renewal times*

As an example we investigate the case of the Gamma(λ, r) distribution. For this case we have

$$\mu = \frac{r}{\lambda}, \ \mu^{(2)} = \frac{r(r+1)}{\lambda^2}, \ \mu^{(3)} = \frac{r(r+1)(r+2)}{\lambda^3}$$

Substituting in equation (2.32) we have

$$\lim_{t\to\infty} \mathrm{Cov}[B(t)V(t)] = \frac{\lambda r(r+1)(r+2)}{6r\lambda^3} - \frac{\lambda^2 r^2 (r+1)^2}{4r^2\lambda^4} = \frac{1-r^2}{12\lambda^2}$$

2.8 The Joint Distribution Function of the Number of Renewals and the Sum of the Backward and Forward Recurrence Times

A pioneer work on the sum of the backward and forward recurrence times (which is often referred to as the total life) has been the paper by Skellam and Shenton[39] with a more general viewpoint of the concept. Cohen[8] gives the distribution function of the sum of the backward and forward recurrence times for the stationary renewal process. Feller[12], Heyman and Sobel[21], and Daley and Vere-Jones[10], give a result for the time dependent ordinary renewal process. Here the technique of multiple integrals is applied to derive *the joint distribution function of the number of renewals and the sum of the backward and forward recurrence times* for the time dependent ordinary renewal process. We shall use $S(t)$ to denote the sum of the backward and forward recurrence times; i.e.,

$$S(t) = B(t) + V(t)$$

Again, as was done previously we distinguish two cases.

Case 1. $N(t) = 0$. Clearly then $S(t) = T_1$,

1. For $0 \le s < t$

$$\Pr\{N(t) = 0, S(t) \le s\} = 0 \tag{2.33}$$

2. For $t \le s < \infty$

$$\begin{aligned} \Pr\{N(t) = 0, S(t) \le s\} &= \Pr\{N(t) = 0, V(t) \le s - t\} \\ &= \Pr\{t < T_1 \le s\} \\ &= F(s) - F(t) \end{aligned} \tag{2.34}$$

Case 2. $N(t) = n$ where $n = 1, 2, \ldots$

1. For $0 \le s \le t$

$$\begin{aligned}
&\Pr\{N(t) = n, S(t) \le s\} \\
&= \Pr\{T_1 + T_2 + \cdots + T_n \le t, T_1 + T_2 + \cdots + T_{n+1} > t, \\
&\qquad T_1 + T_2 + \cdots + T_n > t - s, T_{n+1} \le s\} \\
&= \underset{t-s<t_1+\cdots+t_n\le t}{\int\int\cdots\int} f(t_1)f(t_2)\cdots f(t_n) \\
&\qquad \{F(s) - F[t - (t_1 + t_2 + \cdots + t_n)]\} dt_1 dt_2 \ldots dt_n \\
&= \int_{t-s}^{t} f^{*(n)}(u)[F(s) - F(t-u)]du
\end{aligned} \tag{2.35}$$

2. For $t \le s < \infty$

$$\begin{aligned}
&\Pr\{N(t) = n, S(t) \le s\} \\
&= \Pr\{T_1 + T_2 + \cdots + T_n \le t, T_1 + T_2 + \cdots + T_{n+1} > t, T_{n+1} \le s\} \\
&= \underset{0<t_1+\cdots+t_n\le t}{\int\int\cdots\int} f(t_1)f(t_2)\cdots f(t_n) \\
&\qquad \{F(s) - F[t - (t_1 + t_2 + \cdots + t_n)]\} dt_1 dt_2 \cdots dt_n \\
&= \int_{0}^{t} f^{*(n)}(u)[F(s) - F(t-u)]du
\end{aligned} \tag{2.36}$$

Thus summarizing we have

$$\Pr\{N(t) = 0, S(t) \le s\} = \begin{cases} 0, & 0 \le s < t \\ F(s) - F(t), & t \le s < \infty \end{cases} \tag{2.37}$$

And for $n = 1, 2, \ldots$

$$\Pr\{N(t) = n, S(t) \le s\} = \begin{cases} \displaystyle\int_{t-s}^{t} f^{*(n)}(u)[F(s) - F(t-u)]du, & 0 \le s < t \\ \displaystyle\int_{0}^{t} f^{*(n)}(u)[F(s) - F(t-u)]du, & t \le s < \infty \end{cases} \tag{2.38}$$

2.9 The Distribution Function of the Sum of the Backward and Forward Recurrence Times

2.9.1 The Time Dependent Case

Combining equations (2.37) and (2.38) we have

$$\Pr\{S(t) \le s\} = \sum_{n=0}^{\infty} \Pr\{N(t) = n, S(t) \le s\}$$

$$
= \begin{cases} \sum_{n=1}^{\infty} \int_{t-s}^{t} f^{*(n)}(u)[F(s) - F(t-u)]du, & 0 \le s < t \\ F(s) - F(t) + \sum_{n=1}^{\infty} \int_{0}^{t} f^{*(n)}(u)[F(s) - F(t-u)]du, & t \le s < \infty \end{cases}
$$

$$
= \begin{cases} \int_{t-s}^{t} m(u)[F(s) - F(t-u)]du, & 0 \le s < t \\ F(s) - F(t) + \int_{0}^{t} m(u)[F(s) - F(t-u)]du, & t \le s < \infty \end{cases}
$$

$$
= \begin{cases} \int_{0}^{s} m(t-v)[F(s) - F(v)]dv, & 0 \le s < t \\ F(s) - F(t) + \int_{0}^{t} m(t-v)[F(s) - F(v)]dv, & t \le s < \infty \end{cases} \tag{2.39}
$$

We can elaborate more on the second part:

$$
\begin{aligned}
& F(s) - F(t) + \int_{0}^{t} m(t-v)[F(s) - F(v)]dv = \\
& = F(s) - F(t) + \int_{0}^{t} m(t-v)F(s)dv - \int_{0}^{t} m(t-v)F(v)dv \\
& = F(s) - F(t) + F(s)M(t) - M(t) + F(t) \\
& = F(s) - M(t)[1 - F(s)]
\end{aligned} \tag{2.40}
$$

Thus we can write

$$
\Pr\{S(t) \le s\} = \begin{cases} \int_{0}^{s} m(t-v)[F(s) - F(v)]dv, & 0 \le s < t \\ F(s) - M(t)[1 - F(s)], & t \le s < \infty \end{cases} \tag{2.41}
$$

Note that the second part of equation (2.41) implies that

$$
F(t) - M(t)[1 - F(t)] \ge 0 \Rightarrow M(t) \le \frac{F(t)}{1 - F(t)}
$$

2.9.2 The Limit of the Distribution Function of $S(t)$ as $t \to \infty$

The limit of the sum of the backward and forward recurrence times is obtained if we let $t \to \infty$ in equation (2.41).

$$
\begin{aligned}
\lim_{t\to\infty} \Pr\{S(t) \le s\} &= \lim_{t\to\infty} \int_{0}^{s} m(t-v)[F(s) - F(v)]dv \\
&= \frac{1}{\mu} \int_{0}^{s} [F(s) - F(v)]dv = \frac{1}{\mu} \int_{0}^{s} \int_{v}^{s} dF(u)dv \\
&= \frac{1}{\mu} \int_{0}^{s} \int_{0}^{u} dv dF(u) \\
&= \frac{1}{\mu} \int_{0}^{s} u dF(u)
\end{aligned} \tag{2.42}
$$

This result is also presented by Cohen[8].

Example 4 *The Poisson process*

Again as an example we investigate the case of the negative exponential inter-renewal times as described in Example 1.

Substituting in equation (2.41), we obtain

$$\Pr\{S(t) \leq s\} = \begin{cases} \int_0^s \lambda(e^{-\lambda v} - e^{-\lambda s})dv, & 0 \leq s < t \\ 1 - e^{-\lambda s} - \lambda t(1 - 1 + e^{-\lambda s}), & t \leq s < \infty \end{cases}$$
$$= \begin{cases} 1 - e^{-\lambda s} - \lambda s e^{-\lambda s}, & 0 \leq s < t \\ 1 - e^{-\lambda s} - \lambda t e^{-\lambda s}, & t \leq s < \infty \end{cases}$$

2.10 The Expectation of the Sum of the Backward and Forward Recurrence Times and the Inspection Paradox

2.10.1 The Time Dependent Case

In order to find the expectation of the sum of the backward and forward recurrence times we use the fact that

$$E[S(t)] = \int_0^\infty \Pr\{S(t) > s\}ds$$

First we elaborate more on the first part of equation (2.41):

$$\int_0^s m(t-v)F(s)dv - \int_0^s m(t-v)F(v)dv =$$
$$= F(s)[M(t) - M(t-s)] - M(t) + F(t) + \int_s^t m(t-v)F(v)dv$$
$$= F(s)[M(t) - M(t-s)] - M(t) + F(t) + \int_0^{t-s} m(v)F(t-v)dv \qquad (2.43)$$

Thus,

$$\begin{aligned} \mathrm{E}[S(t)] = &\int_0^t [1 - F(t) + M(t) - F(s)M(t) + F(s)M(t-s) \\ &- \int_0^{t-s} m(v)F(t-v)dv]ds + \int_t^\infty [1 - F(s) + M(t) - F(s)M(t)]ds \\ = &\int_0^t [1 - F(s) + F(s) - F(t) + M(t) - F(s)M(t) + F(s)M(t-s) \\ &- \int_0^{t-s} m(v)F(t-v)dv]ds + \int_t^\infty [1 - F(s) + M(t) - F(s)M(t)]ds \\ = &\int_0^\infty [1 - F(s)]ds + M(t)\int_0^\infty [1 - F(s)]ds + \int_0^t [F(s) - F(t)]ds \\ &+ \int_0^t F(s)M(t-s)ds - \int_0^t \int_0^{t-s} m(v)F(t-v)dvds \end{aligned}$$

$$= \mu + \mu M(t) + \int_0^t [F(s) - F(t)]ds + \int_0^t \int_0^{t-s} m(v)F(s)dv\,ds$$
$$- \int_0^t \int_0^{t-s} m(v)F(t-v)dv\,ds$$
$$= \mu + \mu M(t) - \int_0^t [F(s) - F(t)]ds - \int_0^t \int_0^{t-s} m(v)[F(t-v) - F(s)]dv\,ds$$
$$= \mu + \mu\{F(t) + \int_0^t m(t-v)F(v)dv\}$$
$$- \int_0^t [F(t) - F(s)]ds - \int_0^t \int_0^{t-v} m(v)[F(t-v) - F(s)]dsdv$$
$$= \mu + \mu F(t) + \mu \int_0^t m(t-v)F(v)dv$$
$$- \int_0^t [F(t) - F(s)]ds - \int_0^t m(v) \int_0^{t-v} [F(t-v) - F(s)]dsdv$$
$$= \mu + \mu F(t) + \mu \int_0^t m(t-v)F(u)du - \int_0^t [F(t) - F(s)]ds$$
$$- \int_0^t m(t-u) \int_0^u [F(u) - F(s)]dsdu \tag{2.44}$$

And if we let

$$G(t) = \int_0^t [F(t) - F(s)]ds$$

then the last equation becomes

$$\begin{aligned} \mathrm{E}[S(t)] &= \mu + \mu F(t) - G(t) + \mu m(t) * F(t) - m(t) * G(t) \\ &= \mu + [\mu F(t) - G(t)] + m(t) * [\mu F(t) - G(t)] \end{aligned} \tag{2.45}$$

2.10.2 Generalization of a Well Known Paradox

From Figure 2.1 we can see that $\mu F(t) \geq G(t)$. Therefore it follows immediately that

$$\mathrm{E}[S(t)] \geq \mu \tag{2.46}$$

Equality holds only when $\mu F(t) = G(t)$, which is true for values of time $\{t : F(t) = 0\}$(assuming nondegenerate distributions). For all other cases strict inequality holds.

Let us see the implications of this inequality. Suppose that one desires to estimate the mean inter-renewal time of a renewal process. He may decide to do so by inspecting several realizations of this process, at a certain time epoch and measuring the sum of the backward and forward recurrence times. Then this result will be biased since the mean of this sum will be greater than the actual mean inter-renewal time. This result appears to be a paradox. It

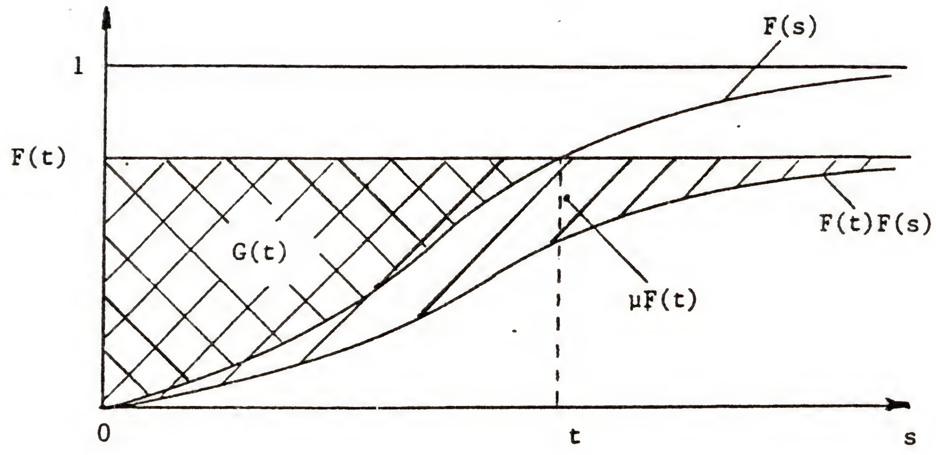

Figure 2.1. Comparison between $G(t)$ and $\mu F(t)$

has been known in the literature as the *inspection paradox* and extensive discussion for it has been made in the intruduction, where an explanation was provided (see subsection 1.6.7). It appears to have been investigated only for the steady state case (e.g. see Feller[12], Karlin and Taylor[24], Heyman and Sobel[21] and Daley and Vere-Jones[10]. Karlin and Taylor[24] also investigated the time dependent case for the negative exponential distribution. Our result shows that this paradox exists in the more general case of the time dependent problem and arbitrary distribution of inter-renewal times with the stated proviso.

2.10.3 The Limit as $t \to \infty$

From equation (2.45) it follows immediately that

$$\begin{aligned}\lim_{t\to\infty} \mathrm{E}[S(t)] &= \lim_{t\to\infty} \{\mu + [\mu F(t) - G(t)] + m(t) * [\mu F(t) - G(t)]\} \\ &= \mu + \frac{1}{\mu}\int_0^\infty [\mu F(v) - G(v)]dv \geq \mu \qquad (2.47)\end{aligned}$$

Example 5 *The Poisson process*

We use equation (2.45) to the case when the inter-renewal intervals obey the negative exponential distribution and recover the results of Karlin and Taylor[24].

$$\mu F(t) = \frac{1}{\lambda}(1 - e^{-\lambda t})$$

$$\begin{aligned} G(t) &= \int_0^t [1 - e^{-\lambda t} - 1 + e^{-\lambda s}]ds \\ &= \int_0^t [e^{-\lambda s} - e^{-\lambda t}]ds = \int_0^t e^{-\lambda s}ds - te^{-\lambda t} \\ &= \frac{1}{\lambda}(1 - e^{-\lambda t}) - te^{-\lambda t} \end{aligned}$$

Therefore

$$\mu F(t) - G(t) = \frac{1}{\lambda}(1 - e^{-\lambda t}) - \frac{1}{\lambda}(1 - e^{-\lambda t}) + te^{-\lambda t} = te^{-\lambda t}$$

And substituting into equation (2.45) we get

$$\begin{aligned} \mathrm{E}[S(t)] &= \mu + te^{-\lambda t} + \int_0^t \lambda v e^{-\lambda v} dv \\ &= \mu + te^{-\lambda t} + \frac{1}{\lambda}(1 - e^{-\lambda t}) - te^{-\lambda t} \\ &= \mu + te^{-\lambda t} + \mu(1 - e^{-\lambda t}) - te^{-\lambda t} \\ &= 2\mu - \mu e^{-\lambda t} \end{aligned}$$

And

$$\lim_{t \to \infty} \mathrm{E}[S(t)] = 2\mu$$

2.11 Conclusion

In this chapter we have complemented existing results on the distribution of the backward and forward recurrence times of an ordinary renewal process. Namely, we obtained novel results for the joint distribution function of the backward and forward recurrence times *and* the number of renewals and for the joint distribution of the sum of backward and forward recurrence times *and* the number of renewals for both the time dependent and the steady state. Hence, we obtained novel results for the covariance of the backward and forward recurrence times and, finally, we generalized for the time dependent case the well-known for the steady state case inspection paradox (see Gakis and Sivazlian[13, 14]).

CHAPTER 3

THE USE OF MULTIPLE INTEGRALS IN THE STUDY OF THE PROBABILITY LAW OF THE RENEWAL COUNTING PROCESS

3.1 Introduction

In this chapter we present several results that complete the characterization of the renewal process. Namely, we study the joint distribution of the number of renewals at three time epochs. Hence, we derive the joint probability law of the renewal increments (for extensive discussion see Section 1.4). Again we derive the results using Theorem 1 of Chapter 1. Sivazlian[36] has used the same methodology in the study of the joint distribution of the number of renewals at two time epochs. Finally, the joint distribution function of two waiting times is used to obtain the *covariance function* of the number of renewals and of the renewal increments.

3.2 The Joint Distribution of the Number of Renewals at Three Time Epochs

3.2.1 Problem Statement

Let $\{X_i, i = 1, 2, \ldots\}$ be a renewal process and $\{N(t), t \geq 0\}$ Consider three time epochs t_1, t_2 and t_3 where $0 \leq t_1 < t_2 < t_3$. Our purpose is to find an expression for $\Pr\{N(t_1) = n_1, N(t_2) = n_2, N(t_3) = n_3\}$ where n_1, n_2, n_3 are nonnegative integers and $0 \leq n_1 \leq n_2 \leq n_3$.

As we did in the previous chapter we derive our results from the multiple integral

$$\int\int_{\mathcal{R}}\cdots\int f(x_1)f(x_2)\cdots f(x_n)[1 - F(t - x_1 - x_2 - \cdots - x_n)]dx_1\, dx_2 \ldots dx_n$$

where $\mathcal{R}$ is the region of integration that describes the desired event.

3.2.2 Evaluation of $\Pr\{N(t_1) = n_1, N(t_2) = n_2, N(t_3) = n_3\}$

For the cases that the problem of evaluation of $\Pr\{N(t_1) = n_1, N(t_2) = n_2, N(t_3) = n_3\}$ reduces to a problem of evaluating $\Pr\{N(t_2) = n_2, N(t_3) = n_3\}$ we use the results from

Sivazlian[36]. In all cases examined we assume $0 \leq t_1 < t_2 < t_3$.

1) Evaluation of $\Pr\{N(t_1) = 0, N(t_2) = 0, N(t_3) = 0\}$. Clearly in this case

$$\begin{aligned} \Pr\{N(t_1) = 0, N(t_2) = 0, N(t_3) = 0\} &= \Pr\{N(t_3) = 0\} \\ &= 1 - F(t_3) \end{aligned} \tag{3.1}$$

2) Evaluation of $\Pr\{N(t_1) = 0, N(t_2) = 0, N(t_3) = 1\}$. Clearly in this case

$$\Pr\{N(t_1) = 0, N(t_2) = 0, N(t_3) = 1\}$$

$$\Pr\{N(t_2) = 0, N(t_3) = 1\}$$

$$= \int_{t_2}^{t_3} f(x)[1 - F(t_3 - x)]dx \tag{3.2}$$

3) Evaluation of $\Pr\{N(t_1) = 0, N(t_2) = 0, N(t_3) = n\}$ where $n \geq 2$. Clearly in this case

$$\Pr\{N(t_1) = 0, N(t_2) = 0, N(t_3) = 1\} = \Pr\{N(t_2) = 0, N(t_3) = n\}$$

$$= \int_{t_2}^{t_3} f(w) \int_0^{t_3 - w} f^{*(n-1)}(x)[1 - F(t_3 - x - w)]dx\, dw \tag{3.3}$$

4) Evaluation of $\Pr\{N(t_1) = 0, N(t_2) = 1, N(t_3) = 1\}$. Clearly in this case

$$\Pr\{N(t_1) = 0, N(t_2) = 1, N(t_3) = 1\} = \int_{\mathcal{R}} f(x_1)[1 - F(t_3 - x_1 - x_2)]dx_1$$

where

$$\mathcal{R} = \{\mathbf{x} : t_1 < x_1 \leq t_2\}$$

and thus

$$\Pr\{N(t_1) = 0, N(t_2) = 1, N(t_3) = 1\} = \int_{t_1}^{t_2} f(x)[1 - F(t_3 - x)]dx \tag{3.4}$$

5) Evaluation of $\Pr\{N(t_1) = 0, N(t_2) = 1, N(t_3) = 2\}$. Clearly in this case

$$\Pr\{N(t_1) = 0, N(t_2) = 1, N(t_3) = 2\}$$

$$= \int \int_{\mathcal{R}} f(x_1) f(x_2)[1 - F(t_3 - x_1 - x_2)]dx_1\, dx_2 \ldots dx_n$$

where

$$\begin{aligned}\mathcal{R} &= \{\mathbf{x} : t_1 < x_1 \leq t_2, t_2 < x_1 + x_2 \leq t_3\} \\ &= \{\mathbf{x} : t_1 < x_1 \leq t_2, t_2 - x_1 < x_2 \leq t_3 - x_1\}\end{aligned}$$

and thus

$$\begin{aligned}&\Pr\{N(t_1) = 0, N(t_2) = 1, N(t_3) = 2\} \\ &= \int_{t_1}^{t_2} f(w) \int_{t_2-w}^{t_3-w} f(x)[1 - F(t_3 - x - w)]dx\, dw \qquad (3.5)\end{aligned}$$

6) Evaluation of $\Pr\{N(t_1) = 0, N(t_2) = 1, N(t_3) = n\}$ where $n \geq 3$. Clearly in this case

$$\begin{aligned}&\Pr\{N(t_1) = 0, N(t_2) = 1, N(t_3) = n\} \\ &= \int\int_{\mathcal{R}} \cdots \int f(x_1)f(x_2)\cdots f(x_n)[1 - F(t_3 - x_1 - x_2 - \cdots - x_n)]dx_1\, dx_2 \cdots dx_n\end{aligned}$$

where

$$\mathcal{R} = \{\mathbf{x} : t_1 < x_1 \leq t_2, t_2 < x_1 + x_2 \leq t_3, t_2 < x_1 + x_2 + \cdots - x_n \leq t_3\}$$

or equivalently

$$\begin{aligned}\mathcal{R} = \{&\mathbf{x} : t_1 < x_1 \leq t_2, t_2 - x_1 < x_2 \leq t_3 - x_1, \\ &0 < x_3 + \cdots + x_n \leq t_3 - x_1 - x_2\}\end{aligned}$$

and thus

$$\begin{aligned}&\Pr\{N(t_1) = 0, N(t_2) = 1, N(t_3) = 2\} \\ &= \int_{t_1}^{t_2} f(v) \int_{t_2-v}^{t_3-v} f(w) \int_0^{t_3-w-v} f^{*(n-2)}(x) \\ &\qquad [1 - F(t_3 - x - w - v)]dx\, dw\, dv \qquad (3.6)\end{aligned}$$

7) Evaluation of $\Pr\{N(t_1) = 0, N(t_2) = n, N(t_3) = n\}$ where $n \geq 2$. Clearly in this case

$$\begin{aligned}&\Pr\{N(t_1) = 0, N(t_2) = n, N(t_3) = n\} \\ &= \int\int_{\mathcal{R}} \cdots \int f(x_1)f(x_2)\cdots f(x_n) \\ &\qquad [1 - F(t - x_1 - x_2 - \cdots - x_n)]dx_1\, dx_2 \cdots dx_n\end{aligned}$$

where

$$\mathcal{R} \;=\; \{\mathbf{x} : t_1 < x_1 \le t_2, t_1 < x_1 + x_2 + \cdots - x_n \le t_2\}$$

or equivalently

$$\mathcal{R} \;=\; \{\mathbf{x} : t_1 < x_1 \le t_2, 0 < x_2 + \cdots + x_n \le t_2 - x_1\}$$

and thus

$$\begin{aligned}
&\Pr\{N(t_1) = 0, N(t_2) = n, N(t_3) = n\} \\
&= \int_{t_1}^{t_2} f(w) \int_0^{t_2 - w} f^{*(n-1)}(x)[1 - F(t_3 - x - w)]dx\, dw
\end{aligned} \tag{3.7}$$

8) Evaluation of $\Pr\{N(t_1) = 0, N(t_2) = n, N(t_3) = n+1\}$ where $n \ge 2$. Clearly in this case

$$\begin{aligned}
&\Pr\{N(t_1) = 0, N(t_2) = n, N(t_3) = n + 1\} \\
&= \int\int_{\mathcal{R}}\cdots\int f(x_1)f(x_2)\cdots f(x_{n+1}) \\
&\qquad [1 - F(t - x_1 - x_2 - \cdots - x_{n+1})]dx_1\, dx_2 \cdots dx_{n+1}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{R} \;=\; &\{\mathbf{x} : t_1 < x_1 \le t_2, t_1 < x_1 + x_2 + \cdots + x_n \le t_2, \\
&t_2 < x_1 + x_2 + \cdots + x_{n+1} \le t_3\}
\end{aligned}$$

or equivalently

$$\begin{aligned}
\mathcal{R} \;=\; &\{\mathbf{x} : t_1 < x_1 \le t_2, 0 < x_2 + \cdots + x_n \le t_2 - x_1, \\
&t_2 - x_1 - x_2 - \cdots - x_n < x_{n+1} \le t_3 - x_1 - x_2 - \cdots - x_n\}
\end{aligned}$$

and thus

$$\begin{aligned}
&\Pr\{N(t_1) = 0, N(t_2) = n, N(t_3) = n + 1\} \\
&= \int_{t_1}^{t_2} f(v) \int_0^{t_2 - v} f^{*(n-1)}(w) \int_{t_2 - w - v}^{t_3 - w - v} f(x) \\
&\qquad [1 - F(t_3 - x - w - v)]dx\, dw\, dv
\end{aligned} \tag{3.8}$$

9) Evaluation of $\Pr\{N(t_1) = 0, N(t_2) = n, N(t_3) = n_1\}$ where $n \geq 2$ and $n_1 = n + m$, $m \geq 2$. Clearly in this case

$$\begin{aligned}
&\Pr\{N(t_1) = 0, N(t_2) = n, N(t_3) = n_1\} \\
&= \int\int_{\mathcal{R}} \cdots \int f(x_1) f(x_2) \cdots f(x_{n_1})[1 - F(t - x_1 - x_2 - \cdots - x_{n_1})] dx_1\, dx_2 \cdots dx_{n_1}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{R} &= \{\mathbf{x} : t_1 < x_1 \leq t_2, t_1 < x_1 + x_2 + \cdots + x_n \leq t_2, \\
&\quad t_2 < x_1 + x_2 + \cdots + x_{n+1} \leq t_3, t_2 < x_1 + x_2 + \cdots + x_{n_1} \leq t_3\}
\end{aligned}$$

or equivalently

$$\begin{aligned}
\mathcal{R} &= \{\mathbf{x} : t_1 < x_1 \leq t_2, 0 < x_2 + \cdots + x_n \leq t_2 - x_1, \\
&\quad t_2 - x_1 - x_2 - \cdots - x_n < x_{n+1} \leq t_3 - x_1 - x_2 - \cdots - x_n, \\
&\quad 0 \leq x_{n+2} + \cdots + x_{n_1} \leq t_3 - x_1 - x_2 - \cdots - x_{n+1}\}
\end{aligned}$$

and thus

$$\begin{aligned}
&\Pr\{N(t_1) = 0, N(t_2) = n, N(t_3) = n_1\} \\
&= \int_{t_1}^{t_2} f(u) \int_0^{t_2 - u} f^{*(n-1)}(v) \int_{t_2 - v - u}^{t_3 - v - u} f(w) \int_0^{t_3 - w - v - u} f^{*(n_1 - n - 1)}(x) \\
&\qquad [1 - F(t_3 - x - w - v)] dx\, dw\, dv\, du \qquad (3.9)
\end{aligned}$$

10) Evaluation of $\Pr\{N(t_1) = n, N(t_2) = n, N(t_3) = n\}$ where $n \geq 1$. Clearly in this case

$$\begin{aligned}
&\Pr\{N(t_1) = n, N(t_2) = n, N(t_3) = n\} \\
&= \int\int_{\mathcal{R}} \cdots \int f(x_1) f(x_2) \cdots f(x_n)[1 - F(t - x_1 - x_2 - \cdots - x_n)] dx_1\, dx_2 \cdots dx_n
\end{aligned}$$

where

$$\mathcal{R} = \{\mathbf{x} : 0 < x_1 + x_2 + \cdots + x_n \leq t_1\}$$

and thus

$$\Pr\{N(t_1) = n, N(t_2) = n, N(t_3) = n\} = \int_0^{t_1} f^{*(n)}(x)[1 - F(t_3 - x)] dx \qquad (3.10)$$

11) Evaluation of $\Pr\{N(t_1) = n, N(t_2) = n, N(t_3) = n+1\}$ where $n \geq 1$. Clearly in this case

$$\Pr\{N(t_1) = n, N(t_2) = n, N(t_3) = n+1\}$$
$$= \int\int_{\mathcal{R}} \cdots \int f(x_1)f(x_2)\cdots f(x_{n+1})[1 - F(t - x_1 - x_2 - \cdots - x_{n+1})]dx_1\,dx_2\cdots dx_{n+1}$$

where

$$\mathcal{R} = \{\mathbf{x} : 0 < x_1 + x_2 + \cdots + x_n \leq t_1, t_2 < x_1 + x_2 + \cdots + x_{n+1} \leq t_3\} \tag{3.11}$$

or equivalently

$$\mathcal{R} = \{\mathbf{x} : 0 < x_1 + x_2 + \cdots + x_n \leq t_1,$$
$$t_2 - x_1 - x_2 - \cdots - x_n < x_{n+1} \leq t_3 - x_1 - x_2 - \cdots - x_n\}$$

and thus

$$\Pr\{N(t_1) = n, N(t_2) = n, N(t_3) = n\}$$
$$= \int_0^{t_1} f^{*(n)}(w) \int_{t_2-w}^{t_3-w} (x)[1 - F(t_3 - x - w)]dx\,dw \tag{3.12}$$

12) Evaluation of $\Pr\{N(t_1) = n, N(t_2) = n, N(t_3) = n_1\}$ where $n \geq 1$ and $n_1 = n + m$, $m \geq 2$. Clearly in this case

$$\Pr\{N(t_1) = n, N(t_2) = n, N(t_3) = n_1\}$$
$$= \int\int_{\mathcal{R}} \cdots \int f(x_1)f(x_2)\cdots f(x_{n_1})[1 - F(t - x_1 - x_2 - \cdots - x_{n_1})]dx_1\,dx_2\cdots dx_{n_1}$$

where

$$\mathcal{R} = \{\mathbf{x} : 0 < x_1 + x_2 + \cdots + x_n \leq t_1, t_2 < x_1 + x_2 + \cdots + x_{n+1} \leq t_3,$$
$$t_2 < x_1 + x_2 + \cdots + x_{n_1} \leq t_3\}$$

or equivalently

$$\mathcal{R} = \{\mathbf{x} : 0 < x_1 + x_2 + \cdots + x_n \leq t_1,$$
$$t_2 - x_1 - x_2 - \cdots - x_n < x_{n+1} \leq t_3 - x_1 - x_2 - \cdots - x_n,$$
$$0 < x_{n+2} + \cdots + x_{n_1} \leq t_3 - x_1 - x_2 - \cdots - x_{n+1}\}$$

and thus

$$\Pr\{N(t_1) = n, N(t_2) = n, N(t_3) = n_1\}$$
$$= \int_0^{t_1} f^{*(n)}(v) \int_{t_2-v}^{t_3-v} f(w) \int_0^{t_3-v-w} f^{*(n_1-n-1)}(x)$$
$$[1 - F(t_3 - x - w - v)]dx\, dw\, dv \tag{3.13}$$

13) Evaluation of $\Pr\{N(t_1) = n, N(t_2) = n+1, N(t_3) = n+1\}$ where $n \geq 1$. Clearly in this case

$$\Pr\{N(t_1) = n, N(t_2) = n+1, N(t_3) = n+1\}$$
$$= \int\int_{\mathcal{R}} \cdots \int f(x_1)f(x_2)\cdots f(x_{n+1})[1 - F(t - x_1 - x_2 - \cdots - x_{n+1})]dx_1\, dx_2 \cdots dx_{n+1}$$

where

$$\mathcal{R} = \{\mathbf{x} : 0 < x_1 + x_2 + \cdots + x_n \leq t_1, t_1 < x_1 + x_2 + \cdots + x_{n+1} \leq t_2\}$$

or equivalently

$$\mathcal{R} = \{\mathbf{x} : 0 < x_1 + x_2 + \cdots + x_n \leq t_1,$$
$$t_1 - x_1 - x_2 - \cdots - x_n < x_{n+1} \leq t_2 - x_1 - x_2 - \cdots - x_n\}$$

and thus

$$\Pr\{N(t_1) = n, N(t_2) = n+1, N(t_3) = n+1\}$$
$$= \int_0^{t_1} f^{*(n)}(w) \int_{t_1-w}^{t_2-w} f(x)[1 - F(t_3 - x - w)]dx\, dw \tag{3.14}$$

14) Evaluation of $\Pr\{N(t_1) = n, N(t_2) = n+1, N(t_3) = n+2\}$ where $n \geq 1$. Clearly in this case

$$\Pr\{N(t_1) = n, N(t_2) = n+1, N(t_3) = n+2\}$$
$$= \int\int_{\mathcal{R}} \cdots \int f(x_1)f(x_2)\cdots f(x_{n+2})[1 - F(t - x_1 - x_2 - \cdots - x_{n+2})]dx_1\, dx_2 \cdots dx_{n+2}$$

where

$$\mathcal{R} = \{\mathbf{x} : 0 < x_1 + x_2 + \cdots + x_n \leq t_1, t_1 < x_1 + x_2 + \cdots + x_{n+1} \leq t_2,$$
$$t_2 < x_1 + x_2 + \cdots + x_{n+2} \leq t_3\}$$

or equivalently

$$\begin{aligned}
\mathcal{R} &= \{\mathbf{x} : 0 < x_1 + x_2 + \cdots + x_n \le t_1, \\
&\quad t_1 - x_1 - x_2 - \cdots - x_n < x_{n+1} \le t_2 - x_1 - x_2 - \cdots - x_n, \\
&\quad t_2 - x_1 - x_2 - \cdots - x_{n+1} < x_{n+2} \le t_3 - x_1 - x_2 - \cdots - x_{n+1}\}
\end{aligned}$$

and thus

$$\begin{aligned}
&\Pr\{N(t_1) = n, N(t_2) = n+1, N(t_3) = n+2\} \\
&= \int_0^{t_1} f^{*(n)}(v) \int_{t_1-v}^{t_2-v} f(w) \int_{t_2-v-w}^{t_3-v-w} f(x)[1 - F(t_3 - x - w)]dx\, dw\, dv \qquad (3.15)
\end{aligned}$$

15) Evaluation of $\Pr\{N(t_1) = n, N(t_2) = n+1, N(t_3) = n_1\}$ where $n \ge 1$ and $n_1 = n+1+m$, $m \ge 2$. Clearly in this case

$$\begin{aligned}
&\Pr\{N(t_1) = n, N(t_2) = n+1, N(t_3) = n_1\} = \\
&= \int\int_{\mathcal{R}} \cdots \int f(x_1)f(x_2)\cdots f(x_{n_1})[1 - F(t - x_1 - x_2 - \cdots - x_{n_1})]dx_1\, dx_2 \cdots dx_{n_1}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{R} &= \{\mathbf{x} : 0 < x_1 + x_2 + \cdots + x_n \le t_1, t_1 < x_1 + x_2 + \cdots + x_{n+1} \le t_2, \\
&\quad t_2 < x_1 + x_2 + \cdots + x_{n_1} \le t_3\}
\end{aligned}$$

or equivalently

$$\begin{aligned}
\mathcal{R} &= \{\mathbf{x} : 0 < x_1 + x_2 + \cdots + x_n \le t_1, \\
&\quad t_1 - x_1 - x_2 - \cdots - x_n < x_{n+1} \le t_2 - x_1 - x_2 - \cdots - x_n, \\
&\quad t_2 - x_1 - x_2 - \cdots - x_{n+1} < x_{n+2} \le t_3 - x_1 - x_2 - \cdots - x_{n+1}, \\
&\quad 0 < x_{n+3} + \cdots + x_{n_1} \le t_3 - x_1 - x_2 - \cdots - x_{n+2}\}
\end{aligned}$$

and thus

$$\begin{aligned}
&\Pr\{N(t_1) = n, N(t_2) = n+1, N(t_3) = n+2\} \\
&= \int_0^{t_1} f^{*(n)}(u) \int_{t_1-u}^{t_2-u} f(v) \int_{t_2-u-v}^{t_3-u-v} f(w) \\
&\qquad \int_0^{t_3-u-v-w} f(x)[1 - F(t_3 - x - w)]dx\, dw\, dv\, du \qquad (3.16)
\end{aligned}$$

16) Evaluation of $\Pr\{N(t_1)=n, N(t_2)=n_1, N(t_3)=n_1\}$ where $n \geq 1$ and $n_1 = n+m, m \geq 2$. Clearly in this case

$$\begin{aligned}
&\Pr\{N(t_1)=n, N(t_2)=n_1, N(t_3)=n_1\} \\
&= \int\int_{\mathcal{R}}\cdots\int f(x_1)f(x_2)\cdots f(x_{n_1})[1-F(t-x_1-x_2-\cdots-x_{n_1})]dx_1\,dx_2\cdots dx_{n_1}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{R} &= \{\mathbf{x}: 0 < x_1+x_2+\cdots+x_n \leq t_1, t_1 < x_1+x_2+\cdots+x_{n+1} \leq t_2 \\
&\quad t_1 < x_1+x_2+\cdots+x_{n_1} \leq t_2\}
\end{aligned}$$

or equivalently

$$\begin{aligned}
\mathcal{R} &= \{\mathbf{x}: 0 < x_1+x_2+\cdots+x_n \leq t_1, \\
&\quad t_1-x_1-x_2-\cdots-x_n < x_{n+1} \leq t_2-x_1-x_2-\cdots-x_n, \\
&\quad 0 < x_{n+2}+\cdots+x_{n_1} \leq t_2-x_1-x_2-\cdots-x_{n+1}\}
\end{aligned}$$

and thus

$$\begin{aligned}
&\Pr\{N(t_1)=n, N(t_2)=n_1, N(t_3)=n_1\} \\
&= \int_0^{t_1} f^{*(n)}(v)\int_{t_1-v}^{t_2-v} f(w)\int_0^{t_2-v-w} f^{*(n_1-n-1)}(x) \\
&\qquad [1-F(t_3-x-w-v)]dx\,dw\,dv \qquad (3.17)
\end{aligned}$$

17) Evaluation of $\Pr\{N(t_1)=n, N(t_2)=n_1, N(t_3)=n_1+1\}$ where $n \geq 1$ and $n_1 = n+m, m \geq 2$. Clearly in this case

$$\begin{aligned}
&\Pr\{N(t_1)=n, N(t_2)=n_1, N(t_3)=n_1+1\} \\
&= \int\int_{\mathcal{R}}\cdots\int f(x_1)f(x_2)\cdots f(x_{n_1+1})[1-F(t-x_1-x_2-\cdots-x_{n_1+1})]dx_1\,dx_2\cdots dx_{n_1+1}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{R} &= \{\mathbf{x}: 0 < x_1+x_2+\cdots+x_n \leq t_1, t_1 < x_1+x_2+\cdots+x_{n+1} \leq t_2, \\
&\quad t_1 < x_1+x_2+\cdots+x_{n_1} \leq t_2, t_2 < x_1+x_2+\cdots+x_{n_1} \leq t_3\}
\end{aligned}$$

or equivalently

$$\begin{aligned}\mathcal{R} &= \{\mathbf{x} : 0 < x_1 + x_2 + \cdots + x_n \le t_1, \\ &\quad t_1 - x_1 - x_2 - \cdots - x_n < x_{n+1} \le t_2 - x_1 - x_2 - \cdots - x_n, \\ &\quad 0 < x_{n+2} + \cdots + x_{n_1} \le t_2 - x_1 - x_2 - \cdots - x_{n+1}, \\ &\quad t_2 - x_1 - x_2 - \cdots - x_{n_1} < x_{n_1+1} \le t_3 - x_1 - x_2 - \cdots - x_{n_1}\}\end{aligned}$$

and thus

$$\begin{aligned}&\Pr\{N(t_1) = n, N(t_2) = n_1, N(t_3) = n_1 + 1\} \\ &= \int_0^{t_1} f^{*(n)}(u) \int_{t_1-u}^{t_2-u} f(v) \int_0^{t_2-u-v} f^{*(n_1-n-1)}(w) \\ &\quad \int_{t_2-u-v-w}^{t_3-u-v-w} f(x)[1 - F(t_3 - x - w - v - u)]dx\, dw\, dv\, du \qquad (3.18)\end{aligned}$$

18) Evaluation of $\Pr\{N(t_1) = n, N(t_2) = n_1, N(t_3) = n_2\}$ where $n \ge 1$,$n_1 = n + m$,$m \ge 2$ and $n_2 = n_1 + k, k \ge 2$. Clearly in this case

$$\begin{aligned}&\Pr\{N(t_1) = n, N(t_2) = n_1, N(t_3) = n_2\} \\ &= \int\int_{\mathcal{R}} \cdots \int f(x_1)f(x_2)\cdots f(x_{n_2})[1 - F(t - x_1 - x_2 - \cdots - x_{n_2})]dx_1\, dx_2 \cdots dx_{n_2}\end{aligned}$$

where

$$\begin{aligned}\mathcal{R} &= \{\mathbf{x} : 0 < x_1 + x_2 + \cdots + x_n \le t_1, \\ &\quad t_1 < x_1 + x_2 + \cdots + x_{n+1} \le t_2, t_1 < x_1 + x_2 + \cdots + x_{n_1} \le t_2, \\ &\quad t_2 < x_1 + x_2 + \cdots + x_{n_1+1} \le t_3, t_2 < x_1 + x_2 + \cdots + x_{n_2} \le t_3\}\end{aligned}$$

or equivalently

$$\begin{aligned}\mathcal{R} &= \{\mathbf{x} : 0 < x_1 + x_2 + \cdots + x_n \le t_1, \\ &\quad t_1 - x_1 - x_2 - \cdots - x_n < x_{n+1} \le t_2 - x_1 - x_2 - \cdots - x_n, \\ &\quad 0 < x_{n+2} + \cdots + x_{n_1} \le t_2 - x_1 - x_2 - \cdots - x_{n+1}, \\ &\quad t_2 - x_1 - x_2 - \cdots - x_{n_1} < x_{n_1+1} \le t_3 - x_1 - x_2 - \cdots - x_{n_1} \\ &\quad 0 < x_{n_1+2} + \cdots + x_{n_2} \le t_3 - x_1 - x_2 - \cdots - x_{n_1+1}\}\end{aligned}$$

and thus

$$\begin{aligned}
&\Pr\{N(t_1) = n, N(t_2) = n_1, N(t_3) = n_2\} \\
&= \int_0^{t_1} f^{*(n)}(u) \int_{t_1-u}^{t_2-u} f(v) \int_0^{t_2-u-v} f^{*(n_1-n-1)}(w) \\
&\quad \int_{t_2-u-v-w}^{t_3-u-v-w} f(x) \int_0^{t_3-u-v-w-x} f^{*(n_2-n_1-1)}(y) \\
&\quad [1 - F(t_3 - u - v - w - x - y)]dy\, dx\, dw\, dv\, du
\end{aligned} \tag{3.19}$$

Alternatively, we have if we let

$$f^{*(0)}(t) = u_0(t)$$

where u_0 is the unit impulse function defined as follows

$$\int_{0^-}^{0^+} u_0(x)dx = 1, \text{ and } \; u_0(t) = 0 \text{ for } \; t \neq 0$$

we can write the expression for $\Pr\{N(t_1) = n_1, N(t_2) = n_2, N(t_3) = n_3\}$ in a more compact form:

1. For $n = 0, 1, 2, \ldots$

$$\Pr\{N(t_1) = n, N(t_2) = n, N(t_3) = n\} = \int_0^{t_1} f^{*(n)}(y)[1 - F(t_3 - y)]dy \tag{3.20}$$

2. For $n = 0, 1, 2, \ldots$ and $n_1 = n + m, m = 1, 2, \ldots$

$$\begin{aligned}
&\Pr\{N(t_1) = n, N(t_2) = n, N(t_3) = n_1\} \\
&= \int_0^{t_1} f^{*(n)}(w) \int_{t_2-w}^{t_3-w} f(x) \int_0^{t_3-w-x} f^{*(n_1-n-1)}(y) \\
&\quad [1 - F(t_3 - w - x - y)]dy\, dx\, dw \\
&= \int_0^{t_1} f^{*(n)}(w) \int_{t_2-w}^{t_3-w} f(x) \int_0^{t_3-w-x} f^{*(m-1)}(y) \\
&\quad [1 - F(t_3 - w - x - y)]dy\, dx\, dw
\end{aligned} \tag{3.21}$$

3. For $n = 0, 1, 2, \ldots$ and $n_1 = n + m$, $m = 1, 2, \ldots$

$$\Pr\{N(t_1) = n, N(t_2) = n_1, N(t_3) = n_1\}$$

$$= \int_0^{t_1} f^{*(n)}(w) \int_{t_1-w}^{t_2-w} f(x) \int_0^{t_2-w-x} f^{*(n_1-n-1)}(y)$$
$$[1 - F(t_3 - w - x - y)]dy\,dx\,dw$$
$$= \int_0^{t_1} f^{*(n)}(w) \int_{t_1-w}^{t_2-w} f(x) \int_0^{t_2-w-x} f^{*(m-1)}(y)$$
$$[1 - F(t_3 - w - x - y)]dy\,dx\,dw \tag{3.22}$$

4. For $n = 0, 1, 2, \ldots$, $n_1 = n + m$, $m = 1, 2, \ldots$ and $n_2 = n_1 + k$, $k = 1, 2, \ldots$

$$\Pr\{N(t_1) = n, N(t_2) = n_1, N(t_3) = n_2\} =$$
$$= \int_0^{t_1} f^{*(n)}(u) \int_{t_1-u}^{t_2-u} f(v) \int_0^{t_2-u-v} f^{*(n_1-n-1)}(w)$$
$$\int_{t_2-u-v-w}^{t_3-u-v-w} f(x) \int_0^{t_3-u-v-w-x} f^{*(n_2-n_1-1)}(y)$$
$$[1 - F(t_3 - u - v - w - x - y)]dy\,dx\,dw\,dv\,du$$
$$= \int_0^{t_1} f^{*(n)}(u) \int_{t_1-u}^{t_2-u} f(v) \int_0^{t_2-u-v} f^{*(m-1)}(w)$$
$$\int_{t_2-u-v-w}^{t_3-u-v-w} f(x) \int_0^{t_3-u-v-w-x} f^{*(k-1)}(y)$$
$$[1 - F(t_3 - u - v - w - x - y)]dy\,dx\,dw\,dv\,du \tag{3.23}$$

3.3 The Joint Probability Law of the Renewal Increments

The next step is to determine the distribution of the increments i.e.

$$\Pr\{N(t_2) - N(t_1) = m_1, N(t_3) - N(t_2) = m_2\} =$$
$$= \sum_{n_1=1}^{\infty} \Pr\{N(t_1) = n_1, N(t_2) = n_1 + m_1, N(t_3) = n_1 + m_1 + m_2\}$$

and the limit as $t \to \infty$.

3.3.1 The Time Dependent Case

Again,for $0 \le t_1 < t_2 < t_3 < \infty$ we distinguish four cases:

1. For $m_1 = m_2 = 0$

$$\begin{aligned}
\Pr\{N(t_2) - N(t_1) = 0, N(t_3) - N(t_2) = 0\} &= \sum_{n=1}^{\infty} \Pr\{N(t_1) = n, N(t_2) = n, N(t_3) = n\} \\
&= \sum_{n=1}^{\infty} \int_0^{t_1} f^{*(n)}(y)[1 - F(t_3 - y)]dy \\
&= \int_0^{t_1} m(y)[1 - F(t_3 - y)]dy
\end{aligned} \tag{3.24}$$

2. For $m_1 = 0$ and $m_2 = 1, 2, \ldots$

$$\begin{aligned}
&\Pr\{N(t_2) - N(t_1) = 0, N(t_3) - N(t_2) = m_2\} \\
&= \sum_{n=1}^{\infty} \Pr\{N(t_1) = n, N(t_2) = n, N(t_3) = n + m_2\} \\
&= \sum_{n=1}^{\infty} \int_0^{t_1} f^{*(n)}(w) \int_{t_2-w}^{t_3-w} f(x) \int_0^{t_3-w-x} f^{*(m_2-1)}(y) \\
&\qquad [1 - F(t_3 - w - x - y)] dy\, dx\, dw \\
&= \int_0^{t_1} m(w) \int_{t_2-w}^{t_3-w} f(x) \int_0^{t_3-w-x} f^{*(m_2-1)}(y) \\
&\qquad [1 - F(t_3 - w - x - y)] dy\, dx\, dw
\end{aligned} \tag{3.25}$$

3. For $m_1 = 1, 2, \ldots$ and $m_2 = 0$

$$\begin{aligned}
&\Pr\{N(t_2) - N(t_1) = m_1, N(t_3) - N(t_2) = 0\} \\
&= \sum_{n=1}^{\infty} \Pr\{N(t_1) = n, N(t_2) = n + m_1, N(t_3) = n + m_1\} \\
&= \sum_{n=1}^{\infty} \int_0^{t_1} f^{*(n)}(w) \int_{t_1-w}^{t_2-w} f(x) \int_0^{t_2-w-x} f^{*(m_1-1)}(y) \\
&\qquad [1 - F(t_3 - w - x - y)] dy\, dx\, dw \\
&= \int_0^{t_1} m(w) \int_{t_1-w}^{t_2-w} f(x) \int_0^{t_2-w-x} f^{*(m_1-1)}(y) \\
&\qquad [1 - F(t_3 - w - x - y)] dy\, dx\, dw
\end{aligned} \tag{3.26}$$

4. For $m_1 = 1, 2, \ldots$ and $m_2 = 1, 2, \ldots$

$$\begin{aligned}
&\Pr\{N(t_2) - N(t_1) = m_1, N(t_3) - N(t_2) = m_2\} \\
&= \sum_{n=1}^{\infty} \Pr\{N(t_1) = n, N(t_2) = n + m_1, N(t_3) = n + m_1 + m_2\} \\
&= \sum_{n=1}^{\infty} \int_0^{t_1} f^{*(n)}(u) \int_{t_1-u}^{t_2-u} f(v) \int_0^{t_2-u-v} f^{*(m_1-1)}(w) \\
&\qquad \int_{t_2-u-v-w}^{t_3-u-v-w} f(x) = \int_0^{t_3-u-v-w-x} f^{*(m_2-1)}(y) \\
&\qquad [1 - F(t_3 - u - v - w - x - y)] dy\, dx\, dw\, dv\, du \\
&= \int_0^{t_1} m(u) \int_{t_1-u}^{t_2-u} f(v) \int_0^{t_2-u-v} f^{*(m_1-1)}(w)
\end{aligned}$$

$$\int_{t_2-u-v-w}^{t_3-u-v-w} f(x) \int_0^{t_3-u-v-w-x} f^{*(m_2-1)}(y)$$
$$[1 - F(t_3 - u - v - w - x - y)]dy\,dx\,dw\,dv\,du \tag{3.27}$$

3.3.2 The Limit as $t_1 \to \infty$

Using Smith's key renewal theorem we obtain for $t_2 = t_1 + \tau_1$ and $t_3 = t_2 + \tau_2$:

1. For $m_1 = m_2 = 0$

$$\lim_{t_1 \to \infty} \Pr\{N(t_2) - N(t_1) = 0, N(t_3) - N(t_2) = 0\} = \frac{1}{\mu} \int_0^\infty [1 - F(\tau_1 + \tau_2 + y)]dy \tag{3.28}$$

2. For $m_1 = 0$ and $m_2 = 1, 2, \ldots$

$$\lim_{t_1 \to \infty} \Pr\{N(t_2) - N(t_1) = 0, N(t_3) - N(t_2) = m_2\} =$$
$$= \frac{1}{\mu} \int_0^\infty \int_{\tau_1+w}^{\tau_1+\tau_2+w} f(x) \int_0^{\tau_1+\tau_2+w-x} f^{*(m_2-1)}(y)$$
$$[1 - F(\tau_1 + \tau_2 + w - x - y)]dy\,dx\,dw \tag{3.29}$$

3. For $m_1 = 1, 2, \ldots$ and $m_2 = 0$

$$\lim_{t_1 \to \infty} \Pr\{N(t_2) - N(t_1) = m_1, N(t_3) - N(t_2) = 0\}$$
$$= \frac{1}{\mu} \int_0^\infty \int_w^{\tau_1+w} f(x) \int_0^{\tau_1+w-x} f^{*(m_1-1)}(y)$$
$$[1 - F(\tau_1 + \tau_2 + w - x - y)]dy\,dx\,dw \tag{3.30}$$

4. For $m_1 = 1, 2, \ldots$ and $m_2 = 1, 2, \ldots$

$$\lim_{t_1 \to \infty} \Pr\{N(t_2) - N(t_1) = m_1, N(t_3) - N(t_2) = m_2\} =$$
$$= \frac{1}{\mu} \int_0^\infty \int_u^{\tau_1+u} f(v) \int_0^{\tau_1+u-v} f^{*(m_1-1)}(w)$$
$$\int_{\tau_1+u-v-w}^{\tau_1+\tau_2+u-v-w} f(x) \int_0^{\tau_1+\tau_2+u-v-w-x} f^{*(m_2-1)}(y)$$
$$[1 - F(\tau_1 + \tau_2 + u - v - w - x - y)]dy\,dx\,dw\,dv\,du \tag{3.31}$$

3.4 The Covariance Function of the Number of Renewals and of the Renewal Increments

In this section we derive the covariance function of the number of renewals and of the renewal increments in an ordinary renewal counting process. Of course, one could have used the joint distribution of the number of renewals as given in Sivazlian[36] to derive the covariance functions. However, the present methodology provides a much simpler and a more direct approach. First, we write an expression for the joint distribution function of two waiting times which can be easily verified

$$\Pr\{W_k \le y, W_l \le z\} = \int_0^y f^{*(k)}(u) \int_0^{z-u} f^{*(l-k)}(v)\,dv\,du \tag{3.32}$$

Next, we find an expression for $\mathrm{E}[N(t_1)N(t_2)]$ for $0 < t_1 < t_2 < \infty$.

$$\begin{aligned}
\mathrm{E}[N(t_1)N(t_2)] &= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \Pr\{N(t_1) \ge k, N(t_2) \ge l\} \\
&= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \Pr\{W_k \le t_1, W_l \le t_2\} \\
&= \sum_{k=1}^{\infty}\sum_{l=1}^{k} \Pr\{W_k \le t_1\} + \sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty} \Pr\{W_k \le t_1, W_l \le x\} \qquad (3.33)
\end{aligned}$$

We consider each term in the right hand side of (3.33). First

$$\begin{aligned}
\sum_{k=1}^{\infty}\sum_{l=1}^{k} \Pr\{W_k \le t_1\} &= \sum_{k=1}^{\infty} k \Pr\{W_k \le t_1\} \\
&= \sum_{k=1}^{\infty} k \Pr\{N(t_1) \ge k\} \\
&= \sum_{k=1}^{\infty} k \sum_{m=k}^{\infty} \Pr\{N(t_1) = m\} \\
&= \sum_{m=1}^{\infty}\sum_{k=1}^{m} k \Pr\{N(t_1) = m\} \\
&= \sum_{m=1}^{\infty} \frac{m(m+1)}{2} \Pr\{N(t_1) = m\} \\
&= \frac{1}{2}\mathrm{E}[N^2(t_1) + N(t_1)] \qquad (3.34)
\end{aligned}$$

Also, using expression (3.32), the second term on the right hand side of (3.33) becomes

$$\sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty} \Pr\{W_k \le t_1, W_l \le t_2\} = \sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty} \int_0^{t_1} f^{*(k)}(u) \int_0^{t_2-u} f^{*(l-k)}(v)\,dv\,du$$

$$
\begin{aligned}
&= \int_0^{t_1} \sum_{k=1}^{\infty} f^{*(k)}(u) \int_0^{t_2-u} \sum_{r=1} f^{*(r)}(v) dv\, du \\
&= \int_0^{t_1} m(u) \int_0^{t_2-u} m(v) dv\, du \\
&= \int_0^{t_1} m(u) M(t_2 - u) du \qquad (3.35)
\end{aligned}
$$

Thus, substituting (3.34) and (3.35) in (3.33) we obtain

$$
\mathrm{E}[N(t_1)N(t_2)] = \int_0^{t_1} m(u)M(t_2 - u)du + \frac{\mathrm{E}[N^2(t_1)] + \mathrm{E}[N(t_1)]}{2} \qquad (3.36)
$$

We know (see Daley and Vere-Jones[10]) that

$$
\mathrm{E}[N^2(t_1)] = M(t_1) + 2\int_0^{t_1} m(u)M(t_1 - u)du \qquad (3.37)
$$

We can now obtain an expression for $\mathrm{Cov}[N(t_1)N(t_2)]$ where $0 < t_1 \leq t_2 < \infty$

$$
\begin{aligned}
&\mathrm{Cov}[N(t_1)N(t_2)] = \mathrm{E}[N(t_1)N(t_2)] - \mathrm{E}[N(t_1)]\mathrm{E}[N(t_2)] \\
&= \mathrm{E}[N(t_1)N(t_2)] - M(t_1)M(t_2) \qquad (3.38)
\end{aligned}
$$

Substituting (3.37) in (3.36) and using the result in (3.38) yields

$$
\begin{aligned}
\mathrm{Cov}[N(t_1)N(t_2)] &= \int_0^{t_1} m(u)M(t_2 - u)du + \int_0^{t_1} m(u)M(t_1 - u)du + M(t_1) \\
&\quad - M(t_1)M(t_2) \\
&= \int_0^{t_1} m(u)[M(t_2 - u) + M(t_1 - u)]du + M(t_1)[1 - M(t_2)] \qquad (3.39)
\end{aligned}
$$

An expression for $\mathrm{Var}[N(t_1)]$ can immediatelly be obtained by setting $t_1 = t_2$. Thus,

$$
\mathrm{Var}[N(t_1)] = 2\int_0^{t_1} m(u)M(t_1 - u)du + M(t_1) - M^2(t_1) \qquad (3.40)
$$

This result verifies expression (3.37).

We now use expression (3.36) to find an expression for the expectation of the product of the renewal increments, $(0 \leq \tau_1 < t_2 < t_3 < \infty)$

$$
\begin{aligned}
&\mathrm{E}[(N(t_2) - N(t_1))(N(t_3) - N(t_2))] \\
&= \mathrm{E}[N(t_2)N(t_3)] - \mathrm{E}[N^2(t_2)] - \mathrm{E}[N(t_1)N(t_3)] + \mathrm{E}[N(t_1)N(t_2)] \\
&= \int_0^{t_2} m(u)M(t_3 - u)du + \frac{\mathrm{E}[N^2(t_2)] + \mathrm{E}[N(t_2)}{2} - \mathrm{E}[N^2(t_2)]
\end{aligned}
$$

$$- \int_0^{t_1} m(u)M(t_3 - u)du - \frac{\mathrm{E}[N^2(t_1)] + \mathrm{E}[N(t_1)}{2}$$

$$+ \int_0^{t_1} m(u)M(t_2 - u)du + \frac{\mathrm{E}[N^2(t_1)] + \mathrm{E}[N(t_1)}{2}$$

$$= \int_0^{t_2} m(u)M(t_3 - u)du + \int_0^{t_1} m(u)M(t_2 - u)du$$

$$- \int_0^{t_1} m(u)M(t_3 - u)du - \frac{\mathrm{E}[N^2(t_2)] - \mathrm{E}[N(t_2)]}{2}$$

$$= \int_0^{t_1} M(t_2 - u)du + \int_{t_1}^{t_2} m(u)M(t_3 - u)du - \frac{\mathrm{E}[N^2(t_2) - N(t_2)]}{2} \tag{3.41}$$

Therefore, substituting (3.37) in (3.41) we obtain

$$\mathrm{E}[(N(t_2) - N(t_1))(N(t_3) - N(t_2))]$$

$$= \int_0^{t_1} M(t_2 - u)du + \int_{t_1}^{t_2} m(u)M(t_3 - u)du - \int_0^{t_2} m(u)M(t_2 - u)du$$

$$= \int_{t_1}^{t_2} m(u)[M(t_3 - u) - M(t_2 - u)]du \tag{3.42}$$

Additionally,

$$\mathrm{E}[N(t_2) - N(t_1)]\mathrm{E}[N(t_3) - N(t_2)]$$

$$= M(t_2)M(t_3) - M^2(t_2) - M(t_1)M(t_3) + M(t_1)M(t_2)$$

$$= \int_{t_1}^{t_2} m(u)[M(t_3) - M(t_2)]du \tag{3.43}$$

Combining (3.42) and (3.43), we conclude that

$$\mathrm{Cov}[N(t_2) - N(t_1), N(t_3) - N(t_2)]$$

$$= \int_{t_1}^{t_2} m(u)[M(t_3 - u) - M(t_2 - u) - M(t_3) + M(t_2)]du \tag{3.44}$$

which is the required result. Additionally, if we let

$$\tau_1 = t_2 - t_1$$

and

$$\tau_2 = t_3 - t_2 = (t_3 - t_1) - \tau_1$$

expression (3.44) becomes

$$\mathrm{Cov}[N(t_2) - N(t_1), N(t_3) - N(t_2)] =$$

$$= \int_{t_1}^{t_1+\tau_1} m(u)[M(t_1 + \tau_1 + \tau_2 - u) - M(t_1 + \tau_1 - u) - M(t_1 + \tau_1 + d_2) + M(t_1 + \tau_1)]du$$

$$= \int_0^{\tau_1} m(t_1 + v)[M(\tau_1 + \tau_2 - v) - M(\tau_1 - v) - M(t_1 + \tau_1 + \tau_2) + M(t_1 + \tau_1)]dv \tag{3.45}$$

If we let $t_1 \to \infty$, we can obtain the limiting behavior of this covariance expression:

$$
\begin{aligned}
&\lim_{t_1 \to \infty} \mathrm{Cov}[N(t_2) - N(t_1), N(t_3) - N(t_2)] \\
&= \lim_{t_1 \to \infty} \int_0^{\tau_1} m(t_1 + v)[M(\tau_1 + \tau_2 - v) - M(\tau_1 - v) \\
&\quad - M(t_1 + \tau_1 + \tau_2) + M(t_1 + \tau_1)]dv \\
&= \int_0^{\tau_1} \frac{1}{\mu}[M(\tau_1 + \tau_2 - v) - M(\tau_1 - v) - \frac{t_1 + \tau_1 + \tau_2 - t_1 - \tau_1}{\mu}]dv \\
&= \frac{1}{\mu} \int_0^{\tau_1} [M(\tau_2 + w) - M(w) - \frac{\tau_2}{\mu}]dw
\end{aligned}
\tag{3.46}
$$

We can see that

$$
\lim_{t_1 \to \infty} \mathrm{Cov}[N(t_2) - N(t_1), N(t_3) - N(t_2)] = 0
$$

for arbitrary values of τ_1 and τ_2 only when

$$
M(\tau_2 + w) - M(w) - \frac{\tau_2}{\mu} = 0
$$

or equivalently,

$$
M(\tau_2 + w) = M(w) + \frac{\tau_2}{\mu}
$$

This implies that the renewal function $M(\cdot)$ must satisfy the relation:

$$
M(y) = \frac{y}{\mu}
$$

which is true only for the Poisson process.

Although a differential form of the covariance of the number of renewals appears in the recent text by Daley and Vere-Jones[10], it is not amenable to useful expressions. Cohen[8] does not include any specific treatment on the covariance of the number of renewals. The quasi-closed form formulas for the covariance functions just obtained appear to be novel.

3.5 Conclusion

In this chapter we have derived the joint probability law of the number of renewals at three time epochs. Hence we derived the joint probability law of the renewal increments for both the time dependent and the steady state. A closed form expression for the covariance function of the renewal increments was derived and, hence, it was shown that the renewal increments are uncorrelated only for the case when the inter-renewal times are negative exponentially distributed i.e. for the homogeneous Poisson process in which case they are known to be independent.

CHAPTER 4

THE USE OF MULTIPLE INTEGRALS IN THE STUDY OF THE ORDER STATISTICS OF THE ORDINARY RENEWAL PROCESS

4.1 Introduction

The distribution of order statistics in a homogeneous Poisson process where the interarrival times are independent and identically distributed random variables obeying the negative exponential distribution, is well-known and appears in a number of standard references (see e.g Karlin and Taylor[25]). For example it is proved that given the occurence of at least one Poisson event in the interval of time $(0, t)$, the probability of the occurence of any of the events in an infinitessimal interval $d\theta$ is $d\theta/t, 0 < \theta < t$. In other words, the order statistics of the waiting times are ordered samples from a uniform distribution. From these results, the distributions of order statistics are easily derived. When dealing with an ordinary renewal counting process $\{N(t) : t \geq 0\}$ where the inter-renewal times are independent and identically distributed random variables having probability density function $f(\cdot)$ and distribution function $F(\cdot)$ the derivation of the distribution of order statistics has proven difficult and does not seem to have appeared so far in the literature. The difficulty stems from two facts. First, a direct derivation of the joint distribution of waititng times conditioned on $N(t) = n$ renewals does not constitute an appropriate approach for the derivation of the distribution of order statistics. Rather, one has to use and rely on the joint distribution of inter-renewal times *and* the number of renewals to arrive at the required results. The second difficulty stems from the fact that the analytic derivation of these distributions requires the reduction of multiple integral terms, which does not at first seem possible.

However, using Theorem 1 of Chapter 1, we use the multiple integral technique to derive joint and marginal distributions of order statistics of waiting times $\{W_j, j = 1, 2, \ldots\}$ of an ordinary renewal process. In particular, we obtain quasi-closed form expressions for the

distribution of the *minimum*, the *maximum*, the *k-th order statistic* and the *range*. Also, the joint distribution function of *two order statistics* and the *n order statistics* are derived. For each of the derived distributions, the results are used to recover well-known expressions in the special case of the Poisson process.

4.2 Distributions of Order Statistics of Waiting Times

The basis of the present derivations is the joint distribution of the first n inter-renewal times $X_1, X_2, \ldots, X_{N(t)}$ and the number of renewals $N(t)$ or

$$\Pr\{x_1 < X_1 \leq x_1 + dx_1, x_2 < X_2 \leq x_2 + dx_2, \ldots, x_n < X_n \leq x_n + dx_n, N(t) = n\}$$
$$= f(x_1)f(x_2)\ldots f(x_n)[1 - F(t - x_1 - x_2 - \cdots - x_n)]dx_1\, dx_2 \ldots dx_n \tag{4.1}$$

for $n = 1, 2, \ldots$. From this, we can immediately write down the conditional joint probability density function of the first n inter-renewal times given that n ($n \geq 1$) renewals have occurred, i.e.

$$\begin{aligned}
&\Pr\{x_1 < X_1 \leq x_1 + dx_1, x_2 < X_2 \leq x_2 + dx_2, \ldots, x_n < X_n \leq x_n + dx_n | N(t) = n\} \\
&= \frac{\Pr\{x_1 < X_1 \leq x_1 + dx_1, x_2 < X_2 \leq x_2 + dx_2, \ldots, x_n < X_n \leq x_n + dx_n, N(t) = n\}}{\Pr\{N(t) = n\}} \\
&= \frac{f(x_1)f(x_2)\ldots f(x_n)[1 - F(t - x_1 - x_2 - \cdots - x_n)]dx_1\, dx_2 \ldots dx_n}{F^{(n)}(t) - F^{(n+1)}(t)}
\end{aligned} \tag{4.2}$$

For notational simplicity let

$$\Pr\{N(t) = n\} = F^{(n)}(t) - F^{(n+1)}(t) = P_n(t)$$

Expressing equation (4.2) in terms of the waiting times w_i, $i = 1, 2, \ldots, n$, does not appear to be too helpful. For example, we obtain for the joint probability density function of the n waiting times

$$\phi(w_1, w_2, \ldots, w_n) = \frac{1}{P_n(t)} f(w_1)f(w_2 - w_1)\cdots f(w_n - w_{n-1})[1 - F(t - w_n)] \tag{4.3}$$

If we can integrate this expression we can obtain the marginal conditional distribution of the k-th waiting time as follows

$$\begin{aligned}
&\Pr\{W_k \leq w_k | N(t) = n\} \\
&= \frac{1}{P_n(t)} \int_0^{w_k} \cdots \int_0^{w_2} \int_{w_k}^{t} \cdots \int_{w_{n-1}}^{t} f(w_1)f(w_2 - w_1)\cdots f(w_n - w_{n-1}) \\
&\qquad [1 - F(t - w_n)]dw_n \ldots dw_{k+1}\, dw_1 \ldots dw_{k-1}
\end{aligned}$$

The reader can see how difficult it is to manipulate this integral and obtain closed form results. However, applying the technique of multiple integrals we can immediately obtain the desired results in a more useful form using the expression for the joint distribution of the inter-renewal times and the number of renewals.

In each case we identify the region $\mathcal{R}$ of integration of the following integral:

$$\int\int_{\mathcal{R}} \cdots \int f(x_1)f(x_2)\dots f(x_n)[1 - F(t - x_1 - x_2 - \dots - x_n)]dx_1\, dx_2 \dots dx_n \tag{4.4}$$

4.3 The Distribution Function of the First Order Statistic (the Minimum)

We want to find

$$\Pr\{W_1 \leq x | N(t) = n\}$$

First we calculate

$$\begin{aligned} I_1 &= \Pr\{W_1 \leq x, N(t) = n\} \\ &= \int\int_{\mathcal{R}} \cdots \int f(x_1)f(x_2)\dots f(x_n)[1 - F(t - x_1 - x_2 - \dots - x_n)]dx_1\, dx_2 \dots dx_n \end{aligned}$$

for $x < t$ and where

$$\begin{aligned} \mathcal{R} &= \{\mathbf{x} : 0 < x_1 \leq x, 0 < x_1 + x_2 + \dots + x_n \leq t\} \\ &= \{\mathbf{x} : 0 < x_1 \leq x, 0 < x_2 + \dots + x_n \leq t - x_1\} \end{aligned}$$

Thus the integral becomes

$$I_1 = \int_0^x f(x_1) \int\int_{0 < x_2 + \dots + x_n \leq t - x_1} \cdots \int f(x_2)\dots f(x_n)[1 - F(t - x_1 - x_2 - \dots - x_n)]dx_2 \dots dx_n\, dx_1$$

And using Theorem 1

$$\begin{aligned} I_1 &= \int_0^x f(x_1) \int_0^{t-x_1} f^{*(n-1)}(u)[1 - F(t - x_1 - u)]du\, dx_1 \\ &= \int_0^x f(x_1)[F^{(n-1)}(t - x_1) - F^{(n)}(t - x_1)]dx_1 \end{aligned} \tag{4.5}$$

Therefore, we conclude that

$$\Pr\{W_1 \leq x | N(t) = n\} = \begin{cases} \dfrac{1}{P_n(t)} \displaystyle\int_0^x f(x_1)[F^{(n-1)}(t - x_1) - F^{(n)}(t - x_1)]dx_1, & 0 \leq x < t \\ 1, & t \leq v < \infty \end{cases} \tag{4.6}$$

<u>Example 6</u> *The Poisson process*

As an example we investigate the case when the inter-renewal times obey the negative exponential distribution with parameter λ and consequently mean inter-renewal time $\mu = 1/\lambda$. We remind the reader that this is the special case of the Poisson process. For the negative exponential distribution

$$F(t) = 1 - e^{-\lambda t}, \quad f(t) = \lambda e^{-\lambda t}$$

and for the Poisson process

$$\Pr\{N(t) = n\} = F^{(n)} - F^{(n-1)}(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

Substituting these results into equation (4.6) we obtain

$$\begin{aligned}
&\Pr\{W_1 \le x | N(t) = n\} \\
&= \frac{1}{\frac{(\lambda t)^n e^{-\lambda t}}{n!}} \int_0^x \lambda e^{-\lambda x_1} \frac{[\lambda(t - x_1)]^{n-1} e^{-\lambda(t-x_1)}}{(n-1)!} dx_1 \\
&= \frac{1}{\frac{(\lambda t)^n e^{-\lambda t}}{n!}} \int_0^x \lambda^n e^{-\lambda t} \frac{(t - x_1)^{n-1}}{(n-1)!} dx_1 \\
&= \frac{t^n - (t-x)^n}{t^n} \\
&= 1 - \left(1 - \frac{x}{t}\right)^n, \quad \text{for} \quad 0 \le x < t
\end{aligned}$$

and, of course

$$\Pr\{W_1 \le x | N(t) = n\} = 1, \text{ for } \quad t \le x < \infty$$

<u>4.4 The Distribution Function of the n-th Order Statistic (the maximum)</u>

We want to find

$$\Pr\{W_n \le x | N(t) = n\}$$

First we calculate

$$\begin{aligned}
I_n &= \Pr\{W_n \le x, N(t) = n\} \\
&= \int\!\!\int_{\mathcal{R}} \cdots \int f(x_1) f(x_2) \ldots f(x_n)[1 - F(t - x_1 - x_2 - \cdots - x_n)] dx_1\, dx_2 \ldots dx_n
\end{aligned}$$

for $x < t$ and where

$$\mathcal{R} \;=\; \{\mathbf{x} : 0 < x_1 + x_2 + \cdots + x_n \le x < t\}$$

Thus the integral becomes

$$I_n = \underset{0<x_1+x_2+\cdots+x_n\leq x}{\int\int\cdots\int} f(x_1)f(x_2)\ldots f(x_n)[1-F(t-x_1-x_2-\cdots-x_n)]dx_1\,dx_2\ldots dx_n$$

And using Theorem 1

$$I_n = \int_0^x f^{*(n)}(u)[1-F(t-u)]du$$

Therefore, we conclude

$$\Pr\{W_n \leq x|N(t)=n\} = \begin{cases} \dfrac{1}{P_n(t)}\displaystyle\int_0^x f^{*(n)}(u)[1-F(t-u)]du, & 0\leq x<t \\ 1, & t\leq x<\infty \end{cases} \tag{4.7}$$

<u>Example 7</u> *The Poisson process*

Again we investigate the case when the inter-renewal times obey the negative exponential distribution. Substituting in equation (4.7) we obtain

$$\begin{aligned}
&\Pr\{W_n \leq x|N(t)=n\} \\
&= \frac{1}{\frac{(\lambda t)^n e^{-\lambda t}}{n!}}\int_0^x \frac{\lambda^n u^{n-1}}{(n-1)!}e^{-\lambda u}e^{-\lambda(t-u)}du \\
&= \frac{1}{t^n}\int_0^x nu^{n-1}du \\
&= \frac{x^n}{t^n}, \quad \text{for} \quad 0\leq x<t
\end{aligned}$$

and of course

$$\Pr\{W_n \leq x|N(t)=n\} = 1, \quad \text{for} \quad t\leq x<\infty$$

<u>4.5 The Distribution Function of the k-th Order Statistic</u>

We want to find

$$\Pr\{W_k \leq x|N(t)=n\}, \quad \text{for} \quad 1<k<n$$

First we calculate

$$\begin{aligned}
I_k &= \Pr\{W_k \leq x, N(t)=n\} \\
&= \underset{\mathcal{R}}{\int\int\cdots\int} f(x_1)f(x_2)\ldots f(x_n)[1-F(t-x_1-x_2-\cdots-x_n)]dx_1\,dx_2\ldots dx_n
\end{aligned}$$

for $x<t$ and where

$$\begin{aligned}
\mathcal{R} &= \{\mathbf{x}: 0<x_1+\cdots+x_k\leq x, 0<x_1+x_2+\cdots+x_n\leq t\} \\
&= \{\mathbf{x}: 0<x_1+\cdots+x_k\leq x, 0<x_{k+1}+\cdots+x_n\leq t-x_1-\ldots-x_k\}
\end{aligned}$$

Thus the integral becomes

$$I_k = \iint\cdots\int_{0<x_1+\cdots+x_k\le x} f(x_1)\ldots f(x_k) \iint\cdots\int_{0<x_{k+1}+\cdots+x_n\le t-x_1-\cdots-x_k} f(x_{k+1})\ldots f(x_n)$$
$$[1-F(t-x_1-x_2-\cdots-x_n)]dx_{k+1}\ldots dx_n\,dx_1\ldots dx_k$$

And using Theorem 1

$$\begin{aligned} I_k &= \int_0^x f^{*(k)}(u)\int_0^{t-u} f^{*(n-k)}(v)[1-F(t-v-u)]dv\,du \\ &= \int_0^x f^{*(k)}(u)[F^{(n-k)}(t-u)-F^{(n-k+1)}(t-u)]du \end{aligned}$$

Therefore, we conclude for $1<k<n$ that

$$\Pr\{W_k\le x|N(t)=n\}$$
$$= \begin{cases} \dfrac{1}{P_n(t)}\displaystyle\int_0^x f^{*(k)}(u)[F^{(n-k)}(t-u)-F^{(n-k+1)}(t-u)]du, & 0\le x<t \\ 1, & t\le x<\infty \end{cases} \tag{4.8}$$

We can immediatelly see that this formula also holds for $k=1$ and $k=n$. Although we could have just presented one proof, nevertheless we gave the proofs for the minimum and the maximum to provide the reader with a better insight to the problem.

<u>Example 8</u> *The Poisson process*

Again we investigate the case when the inter-renewal times obey the negative exponential distribution. Substituting in equation (4.8) we obtain for $0\le x<t$

$$\begin{aligned} &\Pr\{W_k\le x|N(t)=n\} \\ &= \frac{1}{\frac{(\lambda t)^n e^{-\lambda t}}{n!}}\int_0^x \frac{\lambda^k u^{k-1}}{(k-1)!}e^{-\lambda u}\frac{[\lambda(t-u)]^{n-k}e^{-\lambda(t-u)}}{(n-k)!}du \\ &= \frac{1}{B(k,n-k+1)}\int_0^x (\frac{u}{t})^{k-1}(1-\frac{u}{t})^{n-k}d\frac{u}{t} \\ &= \frac{1}{B(k,n-k+1)}\int_0^{\frac{x}{t}} w^{k-1}(1-w)^{n-k}dw \end{aligned}$$

which is the well known Beta distribution. Of course for $t\le x<\infty$

$$\Pr\{W_k\le x|N(t)=n\}=1$$

4.6 The Distribution Function of the Range of the Order Statistics

We want to find

$$\Pr\{W_n - W_1 \leq x | N(t) = n\}, 1 < k < n$$

First we calculate for $x < t$

$$\begin{aligned}
I_R &= \Pr\{W_n - W_1 \leq x, N(t) = n\} \\
&= \int\int_{\mathcal{R}} \cdots \int f(x_1)f(x_2)\ldots f(x_n)[1 - F(t - x_1 - x_2 - \cdots - x_n)]dx_1\, dx_2 \ldots dx_n
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{R} &= \{\mathbf{x} : 0 < x_2 + \cdots + x_n \leq x, 0 < x_1 + x_2 + \cdots + x_n \leq t\} \\
&= \{\mathbf{x} : 0 < x_1 \leq t, 0 < x_2 + \cdots + x_n \leq x, 0 < x_2 + \cdots + x_n \leq t - x_1\}
\end{aligned}$$

Thus the integral becomes

$$\begin{aligned}
I_R = \int\int_{0 < x_1 \leq t,} \cdots \int f(x_1) \quad \int\int_{0 < x_2 + \cdots + x_n \leq min[t, t - x_1]} \cdots \int \quad & f(x_2)\ldots f(x_n) \\
& [1 - F(t - x_1 - x_2 - \cdots - x_n)]dx_2 \ldots dx_n\, dx_1
\end{aligned} \tag{4.9}$$

And using Theorem 1

$$\begin{aligned}
I_R &= \int_0^{t-x} f(x_1) \int_0^x f^{*(n-1)}(v)[1 - F(t - v - x_1)]dv\, dx_1 \\
&\qquad + \int_{t-x}^{t} f(x_1) \int_0^{t-x_1} f^{*(n-1)}(u)[1 - F(t - x_1 - u)]du\, dx_1 \\
&= \int_0^{t-x} f(x_1) \int_0^x f^{*(n-1)}(v)[1 - F(t - v - x_1)]dv\, dx_1 \\
&\qquad + \int_{t-x}^{t} f(x_1)[F^{(n-1)}(t - x_1) - F^{(n)}(t - x_1)]dx_1
\end{aligned}$$

Therefore we conclude for $1 < n$

$$\Pr\{W_n - W_1 \leq x | N(t) = n\}$$
$$= \begin{cases} \dfrac{1}{P_n(t)} \displaystyle\int_0^{t-x} f(x_1) \int_0^x f^{*(n-1)}(v)[1 - F(t - v - x_1)]dv\, dx_1 & \\ \quad + \dfrac{1}{P_n(t)} \displaystyle\int_{t-x}^{t} f(x_1)[F^{(n-1)}(t - x_1) - F^{(n)}(t - x_1)]dx_1 & 0 \leq x < t \\ 1, & t \leq x < \infty \end{cases} \tag{4.10}$$

Example 9 *The Poisson process*

Again we investigate the case when the inter-renewal times obey the negative exponential distribution. Substituting in equation (4.10) we obtain

$$
\begin{aligned}
&\Pr\{W_n - W_1 \le x | N(t) = n\} \\
&= \frac{1}{\frac{(\lambda t)^n e^{-\lambda t}}{n!}} \int_0^{t-x} \lambda e^{-\lambda v} \int_0^x \frac{\lambda^{n-1} u^{n-2}}{(n-2)!} e^{-\lambda u} e^{-\lambda(t-u-v)} du\, dv \\
&\quad + \int_{t-x}^{t} \lambda e^{-\lambda v} \frac{[\lambda(t-v)]^{n-1}}{(n-2)!} e^{-\lambda(t-v)} dv \\
&= \frac{(n-1)x^{n-1}t - (n-2)x^n}{t^n}, \quad \text{for} \quad 0 \le x < t
\end{aligned}
$$

and of course

$$\Pr\{W_n \le x | N(t) = n\} = 1, \quad \text{for} \quad t \le x < \infty$$

4.7 The Joint Distribution Function of Two Order Statistics

We want to find for positive integers $k < l \le n$

$$\Pr\{W_k \le y, W_l \le z | N(t) = n\}$$

We consider the following cases for the joint distribution function of W_k, W_l and $N(t)$

1. $0 < y < z < t < \infty$

$$
\begin{aligned}
I_{kl} &= \Pr\{W_k \le y, W_l \le z, N(t) = n\} \\
&= \underset{\mathcal{R}}{\int\int \cdots \int} f(x_1) f(x_2) \ldots f(x_n)[1 - F(t - x_1 - x_2 - \cdots - x_n)] dx_1\, dx_2 \ldots dx_n
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{R} &= \{\mathbf{x} : 0 < x_1 + \cdots + x_k \le y, 0 < x_1 + \cdots + x_l \le z, 0 < x_1 + \cdots + x_n \le t\} \\
&= \{\mathbf{x} : 0 < x_1 + \cdots + x_k \le y, 0 < x_{k+1} + \cdots + x_l \le z - (x_1 + \cdots + x_k), \\
&\quad\; 0 < x_{l+1} + \cdots + x_n \le t - (x_1 + \cdots + x_l)\}
\end{aligned}
$$

Thus the integral becomes

$$
\begin{aligned}
I_{kl} &= \underset{0 < x_1 + \cdots + x_k \le y}{\int\int \cdots \int} f(x_1) \ldots f(x_k) \underset{0 < x_{k+1} + \cdots + x_l \le z - (x_1 + \cdots + x_k)}{\int\int \cdots \int} f(x_{k+1}) \ldots f(x_l) \\
&= \underset{0 < x_{l+1} + \cdots + x_n \le t - (x_1 + \cdots + x_l)}{\int\int \cdots \int} f(x_{l+1}) \ldots f(x_n) \\
&\qquad [1 - F(t - x_1 - x_2 - \cdots - x_n)] dx_1\, dx_2 \ldots dx_n
\end{aligned}
\tag{4.11}
$$

And using Theorem 1

$$I_{kl} = \int_0^y f^{*(k)}(u) \int_0^{z-u} f^{*(l-k)}(v) \int_0^{t-u-v} f^{*(n-l)}(w)[1 - F(t-u-v-w)]dw\,dv\,du$$

$$= \int_0^y f^{*(k)}(u) \int_0^{z-u} f^{*(l-k)}(v) [F^{(n-l)}(t-u-v) - F^{(n-l+1)}(t-u-v)]dv\,du \quad (4.12)$$

2. $0 < y < t \leq z < \infty$

$$\Pr\{W_k \leq y, W_l \leq z, N(t) = n\} = \Pr\{W_k \leq y, N(t) = n\}$$
$$= \int_0^y f^{*(k)}(u)[F^{(n-k)}(t-u) - F^{(n-k+1)}(t-u)]du \quad (4.13)$$

3. $0 < z \leq y < t < \infty$ or $0 < z < t \leq y < \infty$

$$\Pr\{W_k \leq y, W_l \leq z, N(t) = n\} = \Pr\{W_l \leq z, N(t) = n\}$$
$$= \int_0^y f^{*(l)}(u)[F^{(n-l)}(t-u) - F^{(n-l+1)}(t-u)]du \quad (4.14)$$

4. $0 < t \leq y < z < \infty$ or $0 < t \leq z \leq y < \infty$

$$\Pr\{W_k \leq y, W_l \leq z, N(t) = n\} = \Pr\{N(t) = n\}$$
$$= F^{(n)}(t) - F^{(n+1)}(t) \quad (4.15)$$

For the case when $l = n$ we consider again four cases.

1. $0 < y < z < t < \infty$

$$I_{kl} = \Pr\{W_k \leq y, W_l \leq z, N(t) = l\}$$
$$= \iint_{\mathcal{R}} \cdots \int f(x_1)f(x_2)\dots f(x_l)[1 - F(t - x_1 - x_2 - \cdots - x_l)]dx_1\,dx_2\dots dx_l \quad (4.16)$$

where

$$\mathcal{R} = \{\mathbf{x} : 0 < x_1 + \cdots + x_k \leq y, 0 < x_1 + \cdots + x_l \leq z\}$$
$$= \{\mathbf{x} : 0 < x_1 + \cdots + x_k \leq y, 0 < x_{k+1} + \cdots + x_l \leq z - (x_1 + \cdots + x_k)\} \quad (4.17)$$

Thus the integral becomes

$$I_{kl} = \iint_{0<x_1+\cdots+x_k\leq y} \cdots \int f(x_1)\dots f(x_k) \iint_{0<x_{k+1}+\cdots+x_l\leq z-(x_1+\cdots+x_k)} \cdots \int f(x_1)\dots f(x_l)$$
$$[1 - F(t - x_1 - x_2 - \cdots - x_l)]dx_1\,dx_2\dots dx_l \quad (4.18)$$

And using Theorem 1

$$I_{kl} = \int_0^y f^{*(k)}(u) \int_0^{z-u} f^{*(l-k)}(w)[1 - F(t - v - u)]dv\,du \tag{4.19}$$

2. $0 < y < t \leq z < \infty$

$$\begin{aligned}\Pr\{W_k \leq y, W_l \leq z, N(t) = l\} &= \Pr\{W_k \leq y, N(t) = l\} \\ &= \int_0^x f^{*(k)}(u)[F^{(l-k)}(t-u) - F^{(l-k+1)}(t-u)]du \end{aligned} \tag{4.20}$$

3. $0 < z \leq y < t < \infty$ or $0 < z < t \leq y < \infty$

$$\begin{aligned}\Pr\{W_k \leq y, W_l \leq z, N(t) = l\} &= \Pr\{W_l \leq z, N(t) = l\} \\ &= \int_0^y f^{*(l)}(u)[1 - F(t-u)]du \end{aligned} \tag{4.21}$$

4. $0 < t \leq y < z < \infty$ or $0 < t \leq z \leq y < \infty$

$$\begin{aligned}\Pr\{W_k \leq y, W_l \leq z, N(t) = l\} &= \Pr\{N(t) = l\} \\ &= \int_0^t f^{*(l)}(u)[1 - F(t-u)]du \end{aligned} \tag{4.22}$$

Therefore, we conclude for $1 \leq k < l \leq n$

$$\Pr\{W_k \leq y, W_l \leq z | N(t) = n\}$$
$$= \begin{cases} \dfrac{1}{P_n(t)} \displaystyle\int_0^y f^{*(k)}(u) \int_0^{y-u} f^{*(l-k)}(v) & \\ \quad [F^{(n-l)}(t-u-v) - F^{(n-l+1)}(t-u-v)]dv\,du & 0 \leq y < z < t < \infty \\ \dfrac{1}{P_n(t)} \displaystyle\int_0^y f^{*(k)}(u)[F^{(n-k)}(t-u) - F^{(n-k+1)}(t-u)]du, & 0 < y < t \leq z < \infty \\ \dfrac{1}{P_n(t)} \displaystyle\int_0^z f^{*(l)}(u)[F^{(n-l)}(t-u) - F^{(n-l+1)}(t-u)]du, & z < \min\{t, y\} < \infty \\ 1, & t \leq \min\{y, z\} < \infty \end{cases} \tag{4.23}$$

Note that the case when $k = l$ is trivial.

4.8 The Joint Distribution Function of the n Order Statistics

We want to find for $w_1 < w_2 < \ldots < w_n \leq t$

$$\Pr\{W_1 \leq w_1, W_2 \leq w_2, \ldots, W_n \leq w_n, | N(t) = n\},$$

First we calculate

$$\begin{aligned} I_o &= \Pr\{W_1 \leq w_1, W_2 \leq w_2, \ldots W_n \ldots w_n, N(t) = n\}, \\ &= \int\int_{\mathcal{R}} \cdots \int f(x_1)f(x_2)\ldots f(x_n)[1 - F(t - x_1 - x_2 - \cdots - x_n)]dx_1\,dx_2 \ldots dx_n \end{aligned}$$

where

$$\begin{aligned}\mathcal{R} &= \{\mathbf{x} : 0 < x_1 \le w_1, 0 < x_1 + x_2 \le w_2, \ldots, 0 < x_1 + x_2 + \cdots + x_n \le w_n\} \\ &= \{\mathbf{x} : 0 < x_1 \le w_1, 0 < x_2 \le w_2 - x_1, \ldots, 0 < x_n \le w_n - x_1 - x_2 - \ldots - x_{n-1}\}\end{aligned}$$

Thus the integral becomes

$$I_o = \int_0^{w_1} f(x_1) \int_0^{w_2 - x_1} f(x_2) \ldots \int_0^{w_n - x_{n-1} - \ldots - x_1} f(x_n)$$
$$[1 - F(t - x_1 - x_2 - \cdots - x_n)] dx_n \ldots dx_2\, dx_1 \tag{4.24}$$

Therefore, we conclude that for $0 < w_1 < w_2 < \ldots w_n \le t$

$$\Pr\{W_1 \le w_1, W_2 \le w_2, \ldots, W_n \le w_n | N(t) = n\}$$
$$= \frac{1}{P_n(t)} \int_0^{w_1} f(x_1) \int_0^{w_2 - x_1} f(x_2) \ldots \int_0^{w_n - x_{n-1} - \ldots - x_1} f(x_n)$$
$$[1 - F(t - x_1 - x_2 - \cdots - x_n)] dx_n \ldots dx_2\, dx_1 \tag{4.25}$$

Taking the partial derivative with respect to $w_1, w_2, \ldots w_n$, we obtain the conditional joint probability density function of the order statistics which is a result of special importance:

$$\phi(w_1, w_2, \ldots, w_n)$$
$$= \begin{cases} \dfrac{f(w_1) f(w_2 - w_1) \ldots f(w_n - w_{n-1})[1 - F(t - w_n)]}{P_n(t)}, & 0 < w_1 < w_2 < \ldots < w_n \le t \\ 0 & \text{otherwise} \end{cases} \tag{4.26}$$

which is the same with result (4.3).

<u>Example 10</u> *The Poisson process*

Here, using equation (4.26) we recover the well known result for the Poisson process with parameter λ.

$$\begin{aligned}\phi(w_1, w_2, \ldots w_n) &= \frac{\lambda e^{-\lambda w_1} \lambda e^{-\lambda (w_2 - w_1)} \ldots \lambda e^{-\lambda (t - w_n)}}{\frac{(\lambda t)^n e^{-\lambda t}}{n!}} \\ &= \frac{n!}{t^n}\end{aligned}$$

for $w_1 < w_2 < \ldots < w_n \le t$.

4.9 Conclusion

In this chapter we have presented novel results on marginal and joint distribution functions of the order statistics of waiting times of an ordinary renewal process, namely the marginal distribution function of the minimum, the maximum, the k-th order statistic, the range, the joint distribution function of any two order statistics and the joint distribution function of all order statistics (see Gakis and Sivazlian[15]).

CHAPTER 5

EXTENSIONS OF THE RENEWAL PROCESS

5.1 Introduction

We first define the filtered renewal process:

Definition. A stochastic process $\{X(t), t \geq 0\}$ is said to be a *filtered renewal process* if it can be represented by

$$\begin{aligned} X(t) &= \sum_{m=1}^{N(t)} r(t, W_m, Y_m) \\ &= r(t, W_1, Y_1) + r(t, W_2, Y_2) + \cdots + r(t, W_{N(t)}, Y_{N(t)}) \end{aligned} \tag{5.1}$$

where

1. $\{N(t), t \geq 0\}$ is a renewal counting process;
2. $\{Y_m,\ m = 1, 2, \ldots\}$ is a sequence of independently and identically distributed random variables, independent of $\{N(t), t \geq 0\}$;
3. $\{W_m,\ m = 1, 2, \ldots\}$ is a sequence of waiting times of the renewal counting process $\{N(t), t \geq 0\}$;
4. $r(t, x, y)$ is a real valued function of three variables and is called the response function.

Sivazlian[37] presents a result for the characteristic function of the filtered renewal process which we repeat here

$$\begin{aligned} \mathrm{E}[e^{i\sigma X(t)}] &= \mathrm{E}[e^{i\sigma \sum_{m=1}^{N(t)} r(t, W_m, Y_m)}] \\ &= \mathrm{E}[e^{i\sigma \sum_{m=1}^{N(t)} r(t, \sum_{j=1}^{m} T_j, Y_m)}] \end{aligned} \tag{5.2}$$

Conditioning on the number of renewals

$$\mathrm{E}[e^{i\sigma X(t)}] = \sum_{n=0}^{\infty} \mathrm{E}[e^{i\sigma X(t)} | N(t) = n] \Pr\{N(t) = n\} \tag{5.3}$$

If $N(t) = 0$ then $X(t) = 0$ and $\mathrm{E}[e^{i\sigma X(t)}|N(t) = 0] = 1$. Thus, (5.3) can be written

$$\mathrm{E}[e^{\mathrm{i}\sigma X(t)}] = \Pr\{N(t) = 1\} + \sum_{n=0}^{\infty} \mathrm{E}[e^{i\sigma X(t)}|N(t) = n] \Pr\{N(t) = n\}$$

$$= F^c(t) + \sum_{n=1}^{\infty} \int\int_{0<x_1+x_2+\cdots+x_n\leq t}\cdots\int f(x_1)\mathrm{E}[e^{\mathrm{i}\sigma r(t,x_1,Y_1)}]f(x_2)\mathrm{E}[e^{\mathrm{i}\sigma r(t,x_1+x_2,Y_2)}]$$

$$\ldots f(x_n)\mathrm{E}[e^{\mathrm{i}\sigma r(t,x_1+x_2+\cdots+x_n,Y_n)}]F^c(t - x_1 - x_2 - \cdots - x_n)dx_1\, dx_2 \ldots dx_n \tag{5.4}$$

where $F^c(\cdot)$ is the complementary distribution function of the inter-renewal times. If we define

$$\Psi(\sigma, t) = \mathrm{E}[e^{\mathrm{i}\sigma X(t)}] \text{ and } \quad \Xi(\sigma, t, x) = \mathrm{E}[e^{\mathrm{i}\sigma r(t,x,Y)}] \tag{5.5}$$

Then clearly, the integral (5.4) has the form:

$$\Psi(\sigma, t) = F^c(t) + \sum_{n=1}^{\infty} \int\int_{0<x_1+x_2+\cdots+x_n\leq t}\cdots\int \prod_{j=1}^{n} \left[\Xi(\sigma, t, \sum_{l=1}^{j} x_l)f(x_j)\right] F^c(t - \sum_{j=1}^{n} x_j) \prod_{j=1}^{n} dx_j \tag{5.6}$$

The general argument used so far to obtain the characteristic function of the filtered renewal process does not result in a manageable solution.

However, by restricting the form of the response function to make it a function of the difference $t - W_i$, additional results can be obtained. It should be noted however that the forthcoming results apply to a more general class of response *processes*.

5.2 Background Material and Definition of the Extended Compound Renewal Process

A popular class of stochastic processes that appears in the study of many problems in science and engineering is that of the *compound point process* defined as

$$\hat{X}(t) = Y_{01} + Y_{02} + \cdots + Y_{0N(t)}$$

where $\{Y_{0i}, i = 1, 2, \ldots\}$ is a sequence of independently and identically distributed random variables and $\{N(t), t \geq 0\}$ is a point process. The compound point process is, thus, a sum of a *random* number of independently and identically distributed random variables. An extension of the compound point process that appears in the literature is the *compound stream* defined as

$$\hat{X}(t) = Y_1(t) + Y_2(t) + \cdots + Y_{N(t)}(t)$$

where $\{Y_i(t), t \geq 0,\ i = 1, 2, \ldots\}$ is now a sequence of independently and identically distributed *stochastic processes* with *common origin* at time 0 (see e.g. B. V. Gnedenko and I. N. Kovalenko[18]).

A natural extension of the compound stream would be to allow the origin of the compounded stochastic processes to be *random* and *uncommon*. In this paper we examine the case when the number of the compounded stochastic processes and their origin is determined by an underlying ordinary renewal process.

First we define the notion of the extended compound renewal process as follows A stochastic process $\{X(t), t \geq 0\}$ is said to be an *extended compound renewal process* if it can be represented by

$$\begin{aligned} X(t) &= \sum_{m=1}^{N(t)} Y_m(t - W_m) \\ &= Y_1(t - W_1) + Y_2(t - W_2) + \ldots + Y_{N(t)}(t - W_{N(t)}) \end{aligned} \tag{5.7}$$

where

1. $\{N(t), t \geq 0\}$ is an ordinary renewal process with renewal density function $m(\cdot)$ and a finite mean inter-renewal time μ;

2. $\{Y_m(x), x \geq 0,\ m = 1, 2, \ldots\}$ is a sequence of independently and identically distributed stochastic processes, independent of $\{N(t), t \geq 0\}$ with known statistical characterization; they are called the *response processes*;

3. $\{W_m\}$ is a sequence of waiting times defined as follows

$$W_m = X_1 + X_2 + \ldots + X_m$$

where $\{X_i,\ i = 1, 2, \ldots, m\}$ are the inter-renewal times of the ordinary renewal process $\{N(t), t \geq 0\}$.

The notion of the extended compound renewal process is a generalization of the notion of the filtered Poisson process in the extended sense as defined by Parzen[31] in which the underlying process is assumed to be an ordinary renewal process rather than an homogeneous Poisson process.

We would like to underscore here that although $\{Y_m(x), x \geq 0,\ m = 1, 2, \ldots, N(t)\}$, is a sequence of mutually independent and identically distributed stochastic processes, $\{Y_m(t - W_m), t \geq W_m,\ m = 1, 2, \ldots, N(t)\}$, is a sequence of *strongly dependent* stochastic processes. This dependence makes the study of the statistical properties and behavior of the extended compound renewal process directly from (5.7) prohibitivelly difficult.

Here we derive a second type Volterra integral equation for the characteristic function of the extended compound renewal process (Section 5.3). We solve this integral equation for three special cases and we discuss the solution of the equation for a more general class of underlying renewal processes (Section 5.4). We also obtain a closed form expression for the first moment, and derive a recursive relationship for the higher order *raw* moments (Section 5.5). Moreover, we demonstrate how the $GI^X/G/\infty$ queue can be studied as a special case of the extended compound renewal process. In particular, we recover Liu's et al.[27] result for the expected number of busy channels in the $GI^X/G/\infty$ queue. Additionally, we discuss the queue output and the total backlog for the $GI^X/G/\infty$ queue. And, finally, we give an economic example from telephone traffic (Section 5.6).

5.3 The Characteristic Function of the Extended Compound Renewal Process

The characteristic function of a compound renewal process is the following quantity

$$\mathrm{E}[e^{\mathrm{i}\sigma X(t)}] = \mathrm{E}[e^{\mathrm{i}\sigma \sum_{m=1}^{N(t)} Y(t-W_m)}] \tag{5.8}$$

where without loss of generality the subscript m is dropped from Y_m.

Clearly, for $N(t) = 0$

$$\mathrm{E}[e^{\mathrm{i}\sigma X(t)}] = \mathrm{E}[e^{\mathrm{i}\sigma 0}] = 1$$

Using $f(\cdot)$, $F(\cdot)$ and $F^c(\cdot)$ to denote respectively the probability density function, the distribution function and the complementary distribution function of the inter-renewal times and conditioning on the waiting time of the first renewal, we get

$$\begin{aligned}
\mathrm{E}[e^{\mathrm{i}\sigma X(t)}] &= \mathrm{E}[e^{\mathrm{i}\sigma X(t)} | W_1 > t] \Pr\{W_1 > t\} + \int_0^t \mathrm{E}[e^{\mathrm{i}\sigma X(t)} | W_1 = x] f(x) dx \\
&= F^c(t) + \int_0^t \mathrm{E}[e^{\mathrm{i}\sigma [Y(t-W_1) + \sum_{m=2}^{N(t)} Y(t-W_m)]} | W_1 = x] f(x) dx \\
&= F^c(t) + \int_0^t \mathrm{E}[e^{\mathrm{i}\sigma [Y(t-x) + \sum_{m=1}^{N(t-x)} Y(t-W_m)]}] f(x) dx
\end{aligned}$$

$$= F^c(t) + \int_0^t \mathrm{E}[e^{\mathrm{i}\sigma Y(t-x)}]\mathrm{E}[e^{\mathrm{i}\sigma X(t-x)}]f(x)dx$$
$$= F^c(t) + f(t) * \left\{\mathrm{E}[e^{\mathrm{i}\sigma Y(t)}]\mathrm{E}[e^{\mathrm{i}\sigma X(t)}]\right\} \tag{5.9}$$

So, defining

$$\mathrm{E}[e^{\mathrm{i}\sigma X(t)}] = \Psi(\sigma, t) \text{ and } \mathrm{E}[e^{\mathrm{i}\sigma Y(t)}] = \Xi(\sigma, t) \tag{5.10}$$

the characteristic function of the compound renewal process of the type specified by (5.7) can be found as the solution of an integral equation of the following type.

$$\Psi(\sigma, t) = F^c(t) + f(t) * [\Xi(\sigma, t)\Psi(\sigma, t)] \tag{5.11}$$

This is known to be a second type Volterra integral equation. Volterra type integral equations are known to appear frequently in the study of stochastic processes (e.g. see Tsokos and Padgett[41], and Hromadka and Whitley[22]). A closed form expression for the solution of this type of equations does not seem to have been determined yet. However, for certain classes of inter-renewal time distributions closed form solutions may be obtained. Moreover, from equation (5.11) we can obtain renewal type equations for the higher moments of the extended compound renewal process in terms of the lower moments, assuming that $Y(\cdot)$ has finite moments.

We would like to comment here that the results of this paper can be extended for mixed distributions for inter-renewal times if we write the integral in equation (5.11) as a Stieltjes integral.

A brief note on integral equations. Integral equations frequently arise in the study of the behavior of random variables and stochastic processes. A special case of Volterra integral equation is the renewal equation. Liu, Kashyap and Templeton[27], showed that the generating function of the system size of a $GI^X/G/\infty$ queue is the solution of a second type Volterra integral equation and hence they derived a recursive relationship for the binomial moments. The same approach was applied by Liu and Templeton[28] in the study of the $GR^{X_n}/G_n/\infty$ queue. Our methodology does not restrict itself to discrete processes of which the process of Liu et al.[27] is a very special case. It is applicable to both continuous and discrete processes requiring the use of characteristic functions rather than generating functions.

5.4 Special Cases

5.4.1 Negative Exponentially and Erlang-k Distributed Inter-renewal Times

We now examine the case when the inter-renewal times are negative exponentially distributed i.e. the underlying renewal process is a homogeneous Poisson process so that for $\lambda > 0$, $0 \leq t < \infty$

$$f(t) = \lambda e^{-\lambda t}$$

and

$$F^c(t) = e^{-\lambda t}$$

The Volterra type equation can be written in this case as

$$\Psi(\sigma, t) = e^{-\lambda t} + \int_0^t \lambda e^{-\lambda(t-x)} \Xi(\sigma, x) \Psi(\sigma, x) dx \tag{5.12}$$

which yields after transposing

$$\int_0^t \lambda e^{-\lambda(t-x)} \Xi(\sigma, x) \Psi(\sigma, x) dx = \Psi(\sigma, t) - e^{-\lambda t} \tag{5.13}$$

Taking the derivative with respect to t on both sides of (5.12) we obtain

$$\frac{d\Psi(\sigma, t)}{dt} = -\lambda e^{-\lambda t} - \lambda \int_0^t \lambda e^{-\lambda(t-x)} \Xi(\sigma, x) \Psi(\sigma, x) dx + \lambda \Xi(\sigma, t) \Psi(\sigma, t) \tag{5.14}$$

And substituting the expression for the integral (5.13) into (5.14) we obtain

$$\begin{aligned} \frac{d\Psi(\sigma, t)}{dt} &= -\lambda e^{-\lambda t} - \lambda \Psi(\sigma, t) + \lambda e^{-\lambda t} + \lambda t \Xi(\sigma, t) \Psi(\sigma, t) \\ &= -\lambda \Psi(\sigma, t)[1 - \Xi(\sigma, t)] \end{aligned} \tag{5.15}$$

Hence

$$\frac{d\Psi(\sigma, t)}{\Psi(\sigma, t)} = -\lambda[1 - \Xi(\sigma, t)] dt \tag{5.16}$$

Taking into account the initial condition $\Psi(\sigma, 0) = 1$, equation (5.16) yields

$$\Psi(\sigma, t) = e^{-\lambda[t - \int_0^t \Xi(\sigma, x) dx]} \tag{5.17}$$

This result appears in Parzen[31] in his study of the filtered Poisson process (in the extended sense).

For Erlang-2 distributed inter-renewal times it can be shown that equation (5.11) can be reduced to a second order linear ordinary differential equation of the following form:

$$\frac{d^2}{dt^2}\Psi(\sigma,t) + 2\lambda\frac{d}{dt}\Psi(\sigma,t) + \lambda^2[1 - \Xi(\sigma,t)]\Psi(\sigma,t) = 0 \tag{5.18}$$

For arbitrary $\Xi(\sigma,t)$, the general solution of equation (5.18) does not seem to fall into one of the classical categories appearing in the literature (see e.g. Ince[23] and Pearson[32]). However, for specific forms of the response processes $\Xi(\sigma,t)$, solutions may be obtainable using one of the standard methods.

In general, for Erlang-k distributed inter-renewal times, $k = 1, 2, \ldots$ it can be shown that (5.11) can be reduced to a k-th order linear ordinary differential equation of the following form:

$$\left(\frac{d}{dt} + \lambda\right)^k \Psi(\sigma,t) - \lambda^k \Xi(\sigma,t)\Psi(\sigma,t) = 0 \tag{5.19}$$

5.4.2 Uniformly Distributed Inter-renewal Times

We now investigate the case of inter-renewal times that are uniformly distributed over $[0,a]$ so that for $a > 0$, $0 \le t < \infty$

$$f(t) = \begin{cases} 1/a & \text{for } 0 \le t \le a \\ 0 & \text{otherwise} \end{cases}$$

and

$$F^c(t) = \begin{cases} 1 & \text{for } -\infty < t \le 0 \\ 1 - t/a & \text{for } 0 \le t \le a \\ 0 & \text{otherwise} \end{cases}$$

The Volterra type equation (5.11) can be written for $0 \le t \le a$

$$\Psi(\sigma,t) = 1 - \frac{t}{a} + \frac{1}{a}\int_0^t \Xi(\sigma,x)\Psi(\sigma,x)dx \tag{5.20}$$

Taking the derivative with respect to t on both sides of (5.20) we obtain

$$\frac{d}{dt}\Psi(\sigma,t) = -\frac{1}{a} + \frac{1}{a}\Xi(\sigma,t)\Psi(\sigma,t) \tag{5.21}$$

The initial condition is

$$\Psi(\sigma,0) = \mathrm{E}[e^{i\sigma X(0)}] = \mathrm{E}[e^{i\sigma 0}] = 1$$

The solution to (5.21) is then

$$\Psi(\sigma,t) = e^{-\frac{1}{a}\int_0^t \Xi(\sigma,y)dy}\left[1 - \frac{1}{a}\int_0^t e^{\frac{1}{a}\int_0^x \Xi(\sigma,z)dz}dx\right] \tag{5.22}$$

For $t = ka + \tau$, $0 < \tau \leq a$, $k = 1, 2, \ldots$ the Volterra type equation can be written

$$\Psi(\sigma, t) = \frac{1}{a} \int_{t-a}^{t} \Xi(\sigma, x) \Psi(\sigma, x) dx \tag{5.23}$$

Taking the derivative with respect to t on both sides of (5.23) we obtain

$$\frac{d}{dt} \Psi(\sigma, t) = \frac{1}{a} \Xi(\sigma, t) \Psi(\sigma, t) - \frac{1}{a} \Xi(\sigma, t-a) \Psi(\sigma, t-a) \tag{5.24}$$

Define

$$\psi_0(\sigma, \tau) = \Psi(\sigma, \tau) \text{ and } \xi_0(\sigma, \tau) = \Xi(\sigma, \tau)$$

and let for $k = 1, 2, \ldots$

$$\psi_k(\sigma, \tau) = \Psi(\sigma, ka + \tau)$$

and

$$\xi_k(\sigma, \tau) = \Xi(\sigma, ka + \tau)$$

Using these quantities into (5.24) we obtain

$$\frac{d}{d\tau} \psi_k(\sigma, \tau) = \frac{1}{a} \xi_k(\sigma, \tau) \psi_k(\sigma, \tau) - \frac{1}{a} \xi_{k-1}(\sigma, \tau) \psi_{k-1}(\sigma, \tau) \tag{5.25}$$

The initial condition is

$$\psi_k(\sigma, 0) = \psi_{k-1}(\sigma, a), \quad k = 1, 2, \ldots$$

The solution to (5.25) is then

$$\psi_k(\sigma, \tau) = e^{-\frac{1}{a} \int_0^\tau \xi_k(\sigma, y) dy} \left[\psi_{k-1}(\sigma, a) - \frac{1}{a} \int_0^\tau \xi_{k-1}(\sigma, x) \psi_{k-1}(\sigma, x) e^{\frac{1}{a} \int_0^x \xi_k(\sigma, z) dz} dx \right] \tag{5.26}$$

The solution is thus obtained recursively.

5.4.3 The Compound Renewal Process

Another case in which equation (5.11) has a readily obtainable solution is when the response process does not depend on t i.e. $\Xi(\sigma, t) = \Xi(\sigma)$. This is the case for the compound renewal process defined as

$$\hat{X}(t) = Y_{01} + Y_{02} + \ldots + Y_{0N(t)}$$

(see also discussion in Section 5.2). Then

$$\Xi(\sigma) = \mathrm{E}[e^{i\sigma Y_0}]$$

In this case equation (5.11) reduces to a renewal equation, which is a Volterra equation of the convolution type,

$$\Psi(\sigma,t) = F^c(t) + \Xi(\sigma)[f(t) * \Psi(\sigma,t)] \tag{5.27}$$

By taking the Laplace transform (assuming their existence) on both sides of (5.27) with respect to t we obtain

$$\mathcal{L}\{\Psi(\sigma,t)\} = \mathcal{L}\{F^c(t)\} + \Xi(\sigma)\mathcal{L}\{f(t)\}\mathcal{L}\{\Psi(\sigma,t)\}$$

Solving for $\mathcal{L}\{\Psi(\sigma,t)\}$

$$\begin{aligned}\mathcal{L}\{\Psi(\sigma,t)\} &= \frac{\mathcal{L}\{F^c(t)\}}{1-\Xi(\sigma)\mathcal{L}\{f(t)\}} \\ &= \frac{1-\mathcal{L}\{f(t\}}{s[1-\Xi(\sigma)\mathcal{L}\{f(t)\}]} \\ &= \frac{1-\mathcal{L}\{f(t)\}}{s}[1+\Xi(\sigma)\mathcal{L}\{f(t)\}+\Xi^2(\sigma)\mathcal{L}\{f(t)\}^2+\cdots]\end{aligned}$$

Inverting

$$\Psi(\sigma,t) = 1 - F(t) + \Xi(\sigma)\int_0^t [1-f(t-x)]f(x)dx + \Xi^2(\sigma)\int_0^t [1-f(t-x)]f^{*(2)}(x)dx + \cdots$$

or equivalently

$$\Psi(\sigma,t) = 1 + [1-\Xi(\sigma)]F(t) + \Xi(\sigma)[1-\Xi(\sigma)]F^{(2)}(t) + \Xi^2(\sigma)[1-\Xi(\sigma)]F^{(3)}(t) + \cdots \tag{5.28}$$

5.4.4 The First Moment of the Extended Compound Renewal Process

We know that

$$\left.\frac{\partial \mathrm{E}[e^{\mathrm{i}\sigma X(t)}]}{\partial\sigma}\right|_{\sigma=0} = \mathrm{i}\mathrm{E}[X(t)] \tag{5.29}$$

and

$$\left.\mathrm{E}[e^{\mathrm{i}\sigma X(t)}]\right|_{\sigma=0} = 1$$

Thus, taking partial derivatives with respect to σ on both sides of (5.9) we obtain

$$\begin{aligned}\frac{\partial}{\partial\sigma}\mathrm{E}[e^{\mathrm{i}\sigma X(t)}] &= 0 + \int_0^t f(x)\frac{\partial}{\partial\sigma}\left\{\mathrm{E}[e^{\mathrm{i}\sigma Y(t-x)}]\mathrm{E}[e^{\mathrm{i}\sigma X(t-x)}]\right\}dx \\ &= \int_0^t f(x)\left\{\mathrm{E}[e^{\mathrm{i}\sigma Y(t-x)}]\frac{\partial}{\partial\sigma}\mathrm{E}[e^{\mathrm{i}\sigma X(t-x)}] + \mathrm{E}[e^{\mathrm{i}\sigma X(t-x)}]\frac{\partial}{\partial\sigma}\mathrm{E}[e^{\mathrm{i}\sigma Y(t-x)}]\right\}dx \end{aligned} \tag{5.30}$$

Let $\sigma = 0$. Then using (5.29) we obtain

$$\begin{aligned}
\mathrm{E}[X(t)] &= \int_0^t f(x)\mathrm{E}[X(t-x)]dx + \int_0^t f(x)\mathrm{E}[Y(t-x)]dx \\
&= f(t) * \mathrm{E}[X(t)] + f(t) * \mathrm{E}[Y(t)] \qquad (5.31)
\end{aligned}$$

Taking the Laplace transform with respect to t on both sides of (5.31) we obtain

$$\mathcal{L}\{\mathrm{E}[X(t)]\} = \mathcal{L}\{f(t)\}\mathcal{L}\{\mathrm{E}[X(t)]\} + \mathcal{L}\{f(t)\}\mathcal{L}\{\mathrm{E}[Y(t)]\} \qquad (5.32)$$

Solving (5.32) with respect to $\mathcal{L}\{\mathrm{E}[X(t)]\}$ we obtain

$$\mathcal{L}\{\mathrm{E}[X(t)]\} = \frac{\mathcal{L}\{f(t)\}}{1-\mathcal{L}\{f(t)\}}\mathcal{L}\{\mathrm{E}[Y(t)]\} \qquad (5.33)$$

We know that an expression for the renewal density function $m(\cdot)$ is given by

$$\mathcal{L}\{m(t)\} = \frac{\mathcal{L}\{f(t)\}}{1-\mathcal{L}\{f(t)\}} \qquad (5.34)$$

Thus substituting (5.34) into (5.33) we obtain

$$\mathcal{L}\{\mathrm{E}[X(t)]\} = \mathcal{L}\{m(t)\}\mathcal{L}\{\mathrm{E}[Y(t)]\} \qquad (5.35)$$

And inverting

$$\mathrm{E}[X(t)] = \int_0^t m(x)\mathrm{E}[Y(t-x)]dx \qquad (5.36)$$

If $\mathrm{E}[Y(t)]$ is a Smith's function then using Smith's key renewal theorem we can determine the limit of the expectation of the extended compound renewal process:

$$\lim_{t\to\infty} \mathrm{E}[X(t)] = \frac{1}{\mu}\int_0^\infty \mathrm{E}[Y(x)]dx \qquad (5.37)$$

5.4.5 Higher Order Moments of the Extended Compound Renewal Process

Similarly, we can obtain higher order raw moments, taking into account that

$$\mathrm{i}^k\mathrm{E}[X^k(t)] = \frac{\partial^k}{\partial\sigma^k}\mathrm{E}[e^{\mathrm{i}\sigma X(t)}]\Big|_{\sigma=0} \qquad (5.38)$$

Thus, assuming that $Y(\cdot)$ has finite moments and taking the $k-$th derivative on both sides of (5.9) we obtain

$$\begin{aligned}
\frac{\partial^k}{\partial\sigma^k}\mathrm{E}[e^{\mathrm{i}\sigma X(t)}]\Big|_{\sigma=0} &= \int_0^t f(x)\frac{\partial^k}{\partial\sigma^k}\left\{\mathrm{E}[e^{\mathrm{i}\sigma Y(t-x)}]\mathrm{E}[e^{\mathrm{i}\sigma X(t-x)}]\right\} dx\Big|_{\sigma=0} \\
= \int_0^t f(x)\sum_{r=0}^{k}\binom{k}{r}\frac{\partial^r}{\partial\sigma^r}\mathrm{E}[e^{\mathrm{i}\sigma Y(t-x)}]&\frac{\partial^{k-r}}{\partial\sigma^{k-r}}\mathrm{E}[e^{\mathrm{i}\sigma X(t-x)}]dx\Big|_{\sigma=0} \qquad (5.39)
\end{aligned}$$

Therefore, using (5.38) we obtain

$$\begin{aligned} \mathrm{E}[X^k(t)] &= \int_0^t f(x) \sum_{r=0}^{k} \binom{k}{r} \mathrm{E}[Y^r(t-x)]\mathrm{E}[X^{k-r}(t-x)]dx \\ &= \int_0^t f(x)\mathrm{E}[X^k(t-x)]dx + \sum_{r=1}^{k} \binom{k}{r} \int_0^t f(x)\mathrm{E}[Y^r(t-x)]\mathrm{E}[X^{k-r}(t-x)]dx \\ &= f(t) * \mathrm{E}[X^k(t)] + \sum_{r=1}^{k} \binom{k}{r} f(t) * \left\{\mathrm{E}[Y^r(t)]\mathrm{E}[X^{k-r}(t)]\right\} \end{aligned} \tag{5.40}$$

Equation (5.40) is a renewal type integral equation. It can be solved similarly to (5.31) by taking its Laplace transform

$$\mathcal{L}\{\mathrm{E}[X^k(t)]\} = \mathcal{L}\{f(t)\}\mathcal{L}\{\mathrm{E}[X^k(t)]\} + \sum_{r=1}^{k} \binom{k}{r} \mathcal{L}\{f(t)\}\mathcal{L}\{\mathrm{E}[Y^r(t)]\mathrm{E}[X^{k-r}(t)]\} \tag{5.41}$$

Solving (5.41) with respect to $\mathcal{L}\{\mathrm{E}[X^k(t)]\}$, we obtain

$$\mathcal{L}\{\mathrm{E}[X^k(t)]\} = \frac{\mathcal{L}\{f(t)\}}{1-\mathcal{L}\{f(t)\}} \sum_{r=1}^{k} \binom{k}{r} \mathcal{L}\{\mathrm{E}[Y^r(t)]\mathrm{E}[X^{k-r}(t)]\} \tag{5.42}$$

Inverting (5.42) we obtain $\mathrm{E}[X^k(t)]$ in terms of the lower moments as

$$\mathrm{E}[X^k(t)] = \sum_{r=1}^{k} \binom{k}{r} \int_0^t m(x)\mathrm{E}[Y^r(t-x)]\mathrm{E}[X^{k-r}(t-x)]dx \tag{5.43}$$

If $\mathrm{E}[Y^k(t)]\mathrm{E}[X^{k-r}(t)]$ is a Smith's function then using Smith's key renewal theorem we can determine the limit of the k-th raw moment of the extended compound renewal process based on the time dependent lower moments:

$$\lim_{t\to\infty} \mathrm{E}[X^k(t)] = \frac{1}{\mu} \sum_{r=1}^{k} \binom{k}{r} \int_0^\infty \mathrm{E}[Y^r(x)]\mathrm{E}[X^{k-r}(x)]dx \tag{5.44}$$

Note that for $k = 1$ we recover the result for the first moment.

5.5 Applications

The extended compound renewal process can be used to generalize existing results for specific applications of the filtered Poisson process. Here, we present four examples of potential applications to operations research.

5.5.1 Application 1. The Expected Number of Busy Channels in the $GI^X/G/\infty$ Queue

Formula (5.36) can be used to derive the expectation of the number of busy channels in the infinite server $GI^X/G/\infty$ queue with random batch arrivals. The $GI^X/G/\infty$ queue

can be considered a special case of the extended renewal compound process whose response process is

$$Y(\tau) = \sum_{j=1}^{B} I(S_j > \tau) \tag{5.45}$$

where, B is a discrete random variable that represents the size of a batch arrival, and S_j is a sequence of nonnegative independent and identically distributed random variables, independent of B, with distribution function $\Phi(\cdot)$ and complementary distribution function $\Phi^c(\cdot)$, representing the service time of the j-th unit of the batch. $I(\cdot)$ is an indicator function defined as follows

$$I(S_j > \tau) = \begin{cases} 1, & \text{if } S_j > \tau \\ 0, & \text{otherwise} \end{cases}$$

Here, as an application of the formulas for the extended compound renewal process we obtain the expected number of busy channels in the $GI^X/G/\infty$ queue. Clearly $Y(\tau)$ is a simple compound process with expectation

$$\begin{aligned} \mathrm{E}[Y(\tau)] &= \mathrm{E}[B]\mathrm{E}[I(S > \tau)] \\ &= \mathrm{E}[B]\Pr\{S > \tau\} \\ &= \mathrm{E}[B][1 - \Phi(\tau)] \\ &= \mathrm{E}[B]\Phi^c(\tau) \end{aligned} \tag{5.46}$$

where without loss of generality the subscript for the service times has been dropped. Substituting (5.46) into (5.36) we obtain

$$\mathrm{E}[X(t)] = \mathrm{E}[B]\int_0^t m(x)\Phi^c(t - x)dx \tag{5.47}$$

and

$$\lim_{t\to\infty} \mathrm{E}[X(t)] = \frac{\mathrm{E}[B]}{\mu}\int_0^\infty \Phi^c(x)dx = \frac{\mathrm{E}[B]\mathrm{E}[S]}{\mu} \tag{5.48}$$

To obtain the Volterra equation for the characteristic function of $X(t)$, we note that

$$\begin{aligned} \Xi(\sigma, t) &= \sum_{b=1}^{\infty} \mathrm{E}[e^{\mathrm{i}\sigma Y(t)}|B = b]\Pr\{B = b\} \\ &= \sum_{b=1}^{\infty} \mathrm{E}[e^{\mathrm{i}\sigma[I(S_1>t)+I(S_2>t)+\ldots+I(S_b>t)]}]\Pr\{B = b\} \\ &= \sum_{b=1}^{\infty} \left\{\mathrm{E}[e^{\mathrm{i}\sigma I(S>t)}]\right\}^b \Pr\{B = b\} \end{aligned} \tag{5.49}$$

The probability generating function of B is defined for $|z| \leq 1$ as

$$G_B(z) = \sum_{b=1}^{\infty} z^b \Pr\{B = b\} \tag{5.50}$$

Also

$$\mathrm{E}[e^{i\sigma I(S>t)}] = \Phi(t) + e^{i\sigma}\Phi^c(t) = 1 - (1 - e^{i\sigma})\Phi^c(t) \tag{5.51}$$

Substituting (5.50) and (5.51) into (5.49), we obtain

$$\Xi(\sigma, t) = G_B\left[1 - (1 - e^{i\sigma})\Phi^c(t)\right] \tag{5.52}$$

Hence the resulting integral equation for the characteristic function is

$$\Psi(\sigma, t) = F^c(t) + f(t) * \left\{G_B\left[1 - (1 - e^{i\sigma})\Phi^c(t)\right]\Psi(\sigma, t)\right\} \tag{5.53}$$

Note that this equation for $\Psi(\sigma, t)$ is defined in terms of the distribution of the inter-arrival times, the probability generating function of the batch size and the distribution of the service time for each batch unit.

Using (5.43) and (5.44) one can recursively obtain the higher moments from the lower order moments of the simple compound process defined by (5.45). We would like to underscore that unlike Liu et al.[27] who provided a recursive relation for the *binomial* moments, our methodology determines a recursive relation for the more general *raw* moments of the system size $X(t)$. Moreover, the methodology of Liu et al.[27] is restricted to discrete stochastic processes, while the present methodology can cover both discrete and continuous stochastic processes. For the more traditional approach to queueing problems with batch arrivals the reader is refered to Chaudhry and Templeton[7].

5.5.2 Application 2. The Output Rate of the $GI^X/G/\infty$ Queue

The *output* $X_0(t)$ of the $GI^X/G/\infty$ can be described by an extended compound renewal process whose response process –using the same notation as that of the previous subsection– is

$$Y_0(\tau) = \sum_{j=1}^{B} I(S_j \leq \tau) \tag{5.54}$$

We note in particular that

$$X_0(t) = \sum_{i=1}^{N(t)} B_i - X(t) \tag{5.55}$$

and

$$\mathrm{E}[X_0(t)] = \mathrm{E}[B]\mathrm{E}[N(t)] - \mathrm{E}[X(t)] \tag{5.56}$$

Thus, $\mathrm{E}[X_0(t)]$ would follow from (5.55) and (5.56). However, expressions for the higher order moments of $X_0(t)$ and for its characteristic function, could not have been readily obtained from (5.55) because of the correlation of $N(t)$ and $X(t)$. The use of the concept of extended compound renewal process as described in the present paper obviates these difficulties.

An expression for $\mathrm{E}[X_0(t)]$ is obtained alternatively as follows: again $Y_0(t)$ is a simple compound process with expected value

$$\begin{aligned}
\mathrm{E}[Y_0(\tau)] &= \mathrm{E}[B]\mathrm{E}[I(S \leq \tau)] \\
&= \mathrm{E}[B]\,\mathrm{Pr}\{S \leq \tau\} \\
&= \mathrm{E}[B]\Phi(\tau)
\end{aligned} \tag{5.57}$$

The expected output becomes then

$$\mathrm{E}[X_0(t)] = \mathrm{E}[B] \int_0^t m(x)\Phi(t-x)dx \tag{5.58}$$

Assuming that S has a probability density function $\phi(\cdot)$, the *output rate* can be obtained as follows

$$\begin{aligned}
\frac{d}{dt}\mathrm{E}[X_0(t)] &= \mathrm{E}[B]\frac{d}{dt}\int_0^t m(x)\Phi(t-x)dx \\
&= \mathrm{E}[B]\int_0^t m(x)\frac{d}{dt}\Phi(t-x)dx + [m(x)\Phi(t-x)]_{x=t} \\
&= \mathrm{E}[B]\int_0^t m(x)\phi(t-x)dx
\end{aligned} \tag{5.59}$$

From Smith's key renewal theorem

$$\lim_{t\to\infty}\frac{d}{dt}\mathrm{E}[X_0(t)] = \frac{\mathrm{E}[B]}{\mu}\int_0^\infty \phi(x)dx = \frac{\mathrm{E}[B]}{\mu} \tag{5.60}$$

which not surprisingly equals the input rate.

Using the same procedure as in subsection 5.6.1, the equation for the characteristic function of $X_0(t)$ is given by

$$\Psi(\sigma,t) = F^c(t) + f(t) * \left\{G_B\left[1-(1-e^{\mathrm{i}\sigma})\Phi(t)\right]\Psi(\sigma,t)\right\} \tag{5.61}$$

5.5.3 Application 3. The Total Backlog in a $GI^X/G/\infty$ Queue

The *total backlog* in a $GI^X/G/\infty$ queue is the total remaining service time that corresponds to the units currently in the system. It can be described by an extended compound renewal process–using the same assumptions and notations as previously–with response process:

$$Y(\tau) = \sum_{i=1}^{B} R_i(\tau)$$

where after dropping the subscript i, the functions $R_i(\tau)$ are defined as

$$R(\tau) = \begin{cases} S - \tau & \text{for } 0 \le \tau \le S \\ 0 & \text{for } S < \tau < \infty \end{cases}$$

Also

$$\mathrm{E}[Y(\tau)] = \mathrm{E}[B]\mathrm{E}[R(\tau)] = \mathrm{E}[B] \int_{\tau}^{\infty} (y - \tau) d\Phi(y) \tag{5.62}$$

Therefore

$$\mathrm{E}[X(t)] = \mathrm{E}[B] \int_0^t m(x) \int_{t-x}^{\infty} [y - (t - x)] d\Phi(y) dx \tag{5.63}$$

From Smith's key renewal theorem

$$\begin{aligned} \lim_{t\to\infty} \mathrm{E}[X(t)] &= \frac{\mathrm{E}[B]}{\mu} \int_0^{\infty} \int_x^{\infty} (y - x) d\Phi(y) dx \\ &= \frac{\mathrm{E}[B]}{\mu} \int_0^{\infty} \int_0^{y} (y - x) dx\, d\Phi(y) \\ &= \frac{\mathrm{E}[B]}{\mu} \int_0^{t} \left[-\frac{(y-x)^2}{2} \right]_{x=0}^{x=y} d\Phi(y) \\ &= \frac{\mathrm{E}[B]}{\mu} \int_0^{\infty} \frac{y^2}{2} d\Phi(y) \\ &= \frac{\mathrm{E}[B]\mathrm{E}[S^2]}{2\mu} \end{aligned} \tag{5.64}$$

The Volterra equation for the characteristic function can be found by using the same procedure as in subsection 5.6.1:

$$\Psi(\sigma, t) = F^c(t) + f(t) * \left\{ G_B \left[1 - \int_t^{\infty} (1 - e^{i\sigma(y-t)}) d\Phi(y) \right] \Psi(\sigma, t) \right\} \tag{5.65}$$

5.5.4 Application 4. Total Expected Revenue from Long Distance Phone Calls

The revenue from a long distance phone call is, usually, a function of the duration of the conversation. Here we assume that for a conversation that started at time 0, the revenue

from that phone call at time τ is:

$$r(\tau, S) = \begin{cases} \alpha\tau & \text{for } 0 \le \tau \le S \\ \alpha S & \text{for } S < \tau < \infty \end{cases}$$

where S is a nonnegative random variable which represents the duration of a conversation, with distribution function $\Phi(\cdot)$ and complementary distribution function $\Phi^c(\cdot)$. Then

$$\begin{aligned} \mathrm{E}[r(\tau, S)] &= \int_0^\infty r(\tau, y)d\Phi(y) \\ &= \int_0^\tau \alpha y d\Phi(y) + \int_\tau^\infty \alpha\tau d\Phi(y) \\ &= \alpha\int_0^\tau y d\Phi(y) + \alpha\tau\Phi^c(\tau) \end{aligned} \tag{5.66}$$

Assuming that the arrival process of long distance phone calls is an ordinary renewal process, then the total revenue $X(t)$ is an extended compound renewal process whose response process is

$$Y(\tau) = r(\tau, S)$$

Therefore, the total revenue $X(t)$ can be written as

$$X(t) = \sum_{j=1}^{N(t)} r(t - W_j, S_j)$$

where the random variable S_j represents the duration of the j-th conversation. Clearly, the total expected revenue at time t is

$$\begin{aligned} \mathrm{E}[X(t)] &= \int_0^t m(x)\mathrm{E}[r(t-x, S)]dx \\ &= \int_0^t m(x)\left\{\alpha\int_0^{t-x} y d\Phi(y) + \alpha(t-x)\Phi^c(t-x)\right\} dx \end{aligned} \tag{5.67}$$

Intuitively we expect that the expected revenue will grow infinitely large as $t \to \infty$ and, thus, we do not investigate the limiting case. It may be interesting, however, to investigate another measure, the *revenue rate* defined as follows

$$\begin{aligned} \tilde{r}(t) &= \frac{d}{dt}\mathrm{E}[X(t)] \\ &= \frac{d}{dt}\int_0^t m(x)\left\{\alpha\int_0^{t-x} y d\Phi(y) + \alpha(t-x)\Phi^c(t-x)\right\} dx \\ &= \int_0^t m(x)\alpha\frac{d}{dt}\left\{\int_0^{t-x} y d\Phi(y) + (t-x)\Phi^c(t-x)\right\} dx \end{aligned}$$

$$
\begin{aligned}
& +\alpha\left[\int_0^{t-x} y d\Phi(y) + (t-x)\Phi^c(t-x)\right]_{x=t} \\
= \ & \alpha \int_0^t m(x)[(t-x)-(t-x)]d\Phi(t-x) + \alpha\int_0^t \Phi^c(t-x)\}dx \\
= \ & \alpha \int_0^t m(x)\Phi^c(t-x)dx \qquad (5.68)
\end{aligned}
$$

Not surprisingly, the same result could have been obtained had we considered an extended compound renewal process with response function:

$$
Y(\tau) = r'(\tau, S) = \frac{d}{d\tau} r(\tau, S) = \begin{cases} \alpha & \text{for } 0 \leq \tau \leq S \\ 0 & \text{for } S < \tau < \infty \end{cases} \qquad (5.69)
$$

Then, the revenue rate at time t is

$$
\begin{aligned}
\tilde{r}(t) &= \int_0^t m(x)\mathrm{E}[r'(t-x, S)]dx \\
&= \int_0^t \alpha m(x)[1-\Phi(t-x)]dx \\
&= \alpha \int_0^t m(x)\Phi^c(t-x)dx \qquad (5.70)
\end{aligned}
$$

Provided that $\mathrm{E}[S]$ exists, we obtain

$$
\tilde{r} = \lim_{t\to\infty} \tilde{r}(t) = \frac{\alpha}{\mu}\int_0^\infty \Phi^c(x)dx = \frac{\alpha \mathrm{E}[S]}{\mu} \qquad (5.71)
$$

Similarly, when the revenue function for one phone call is in general $R(\tau)$, it can be shown that

$$
\tilde{r}(t) = \int_0^t m(x)\Phi^c(t-x)dR(t-x) \qquad (5.72)
$$

and

$$
\tilde{r} = \lim_{t\to\infty} \tilde{r}(t) = \frac{1}{\mu}\int_0^\infty \Phi^c(x)dR(x) \qquad (5.73)
$$

5.6 The Extended Compound Renewal Process as a Special Case of the Extended Compound Point Process

In order to derive the integral equation for the characteristic function of the extended compound renewal process we had to assume that the behavior of the m-th response process depends only on the *difference* of the current observation time and the waiting time of the m-th renewal. We would like to be able, however, to allow the response process to have a

more general form. We generalize the concept of extended compound renewal process by introducing the notion of *extended compound point process*. In this generalized process, we associate with the i-th event occurence of the process $\{N(t), t \geq 0\}$ a stochastic process $\{Y_i(t, W_i), t \geq W_i,\ i = 1, 2, \ldots\}$, characterized by the following:

1. the time origin of $Y_i(t, W_i)$ is the epoch of occurence of the i-th event W_i;

2. the initial condition of $Y_i(t, W_i)$ is $Y_i(W_i, W_i) = Y_{0i}$ a random variable;

3. the random variables Y_{0i}'s are independently and identically distributed conditional on the waiting times W_i;

4. the stochastic processes $\{Y_i(t, W_i), t \geq W_i,\ i = 1, 2, \ldots\}$ are independently and identically distributed conditional on the waiting times W_i.

The *extended compound point process* $\{X(t), t \geq 0\}$ is then defined as

$$\begin{aligned} X(t) &= \sum_{m=1}^{N(t)} Y_m(t, W_m) \\ &= Y_1(t, W_1) + Y_2(t, W_2) + \ldots + Y_{N(t)}(t, W_{N(t)}) \end{aligned} \tag{5.74}$$

In what follows, we shall derive expressions for $\mathrm{E}[X(t)]$ and $\mathrm{Var}[X(t)]$ for the extended compound point process $X(t)$ in terms of the first two moments of $N(t)$ and $Y(t, W_i)$. We also obtain an expression for the higher order moments.

Next we shall apply the results to the $PP^X/G/\infty$ (PP=point process) queue and show how one can derive expressions for the expected number of busy channels and its variance. We shall show that the $PP^X/G/\infty$ queue generalizes existing results for the $M(t)^X/G/\infty$ and the $GI^X/G/\infty$ queues. Again we should note here that deriving results for the *distribution* of the number of busy channels is not possible for the $PP^X/G/\infty$ queue unless one makes additional assumptions concerning the underlying point process $\{N(t), t \geq 0\}$, such as characterizing the properties of its inter-arrival times. Two additional examples related to the $PP/G/\infty$ queue are also provided which include the derivation of the first two moments of the total backlog and of the total revenues from telephone traffic. Finally, we provide an example in which we determine the expectation and the variance of the extended compound point process for specific forms of *empirical* functions for the expectation

and the covariance of the underlying point process. The forms of these functions are in contrast to the ones generated by a renewal or a non-homogeneous Poisson process: they model situations not amenable to treatment by any of these two processes.

Our motivation to study the extended compound point process is the fact that very often, it is possible to provide for a general process such as $\{N(t), t \geq 0\}$ its first two moments. From a practical standpoint, measuring for example E$[N(t)]$, Var$[N(t)]$ and Cov$[N(t_1), N(t_2)]$ through observational data does not appear to be too difficult a task, even without relying on specific assumptions concerning the inter-arrival times. Thus, in a way, it is appealing to rely on the availability of the second order moments to calculate corresponding second order moments of the process $\{X(t), t \geq 0\}$. It is not our purpose here to elaborate on the specific statistical tasks of measuring the first two moments of $\{N(t), t \geq 0\}$. Rather, we may assume that such statistical measures are available, and proceed towards using them to derive expressions for E$[X(t)]$ and Var$[X(t)]$.

5.7 Notations and Basic Relations

In what follows, we introduce the indicating function:

$$I(t, W_i) = \begin{cases} Y(t, W_i) & 0 < W_i \leq t < \infty \\ 0 & \text{otherwise} \end{cases} \tag{5.75}$$

where without loss of generality the subscript i is dropped from $Y_i(t, W_i)$. We also use the concept of double Stieltjes integral defined as follows

Let $\xi(x, y)$ and $\Phi(x, y)$ be two right continuous real-valued functions of $(x, y) \in R^2$. Then

$$\begin{aligned}
&\int_0^t \int_0^t \xi(x, y) \partial^2_{xy} \Phi(x, y) = \\
&= \int_0^t \int_0^t \xi(x, y) \frac{\partial^2}{\partial x \partial y} \Phi(x, y) dx dy + \sum_{x_i \in S_x} \int_0^t \xi(x_i, y) \frac{\partial}{\partial y} \left[\Phi(x_i, y) - \Phi(x_i^-, y) \right] dy \\
&+ \sum_{y_i \in S_y} \int_0^t \xi(x, y_i) \frac{\partial}{\partial x} \left[\Phi(x, y_i) - \Phi(x, y_i^- \right] dx \\
&+ \sum_{(x_i, y_j) \in S_{xy}} \sum \xi(x_i, y_j) \left[\Phi(x_i, y_j) - \Phi(x_i^-, y_j) - \Phi(x_i, y_j^-) + \Phi(x_i^-, y_j^-) \right]
\end{aligned} \tag{5.76}$$

where S_x, S_y, S_{xy} are the sets of points of nondifferentiability of $\Phi(x, y)$ with respect to x, y and (x, y) respectively.

We also recall that

1.

$$\Pr\{W_m \leq x\} = \Pr\{N(x) \geq m\} \tag{5.77}$$

2.

$$\begin{aligned} \mathrm{E}[N(x)] &= \sum_{m=1}^{\infty} m \Pr\{N(x) = m\} \\ &= \sum_{m=1}^{\infty} \Pr\{N(x) \geq m\} = \sum_{m=1}^{\infty} \Pr\{W_m \leq x\} \end{aligned} \tag{5.78}$$

3. Similarly

$$\begin{aligned} \mathrm{E}[N(x_1)\ldots N(x_r)] &= \sum_{l_1=1}^{\infty} \ldots \sum_{l_k=1}^{\infty} \Pr\{N(x_1) \geq l_1, \ldots, N(x_r) \geq l_k\} \\ &= \sum_{l_1=1}^{\infty} \ldots \sum_{l_k=1}^{\infty} \Pr\{W_{l_1} \leq x_1, \ldots, W_{l_r} \leq x_k\} \end{aligned} \tag{5.79}$$

In particular for $x < y$

$$\begin{aligned} &\mathrm{E}[N(x)N(y)] = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \Pr\{N(x) \geq m, N(y) \geq k\} \\ &= \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \Pr\{W_m \leq x, W_k \leq y\} \\ &= \sum_{m=1}^{\infty} \sum_{k=1}^{m} \Pr\{W_m \leq x, W_k \leq y\} \\ &+ \sum_{m=1}^{\infty} \sum_{k=m+1}^{\infty} \Pr\{W_m \leq x, W_k \leq y\} \end{aligned} \tag{5.80}$$

Also, for $k \leq m$

$$\Pr\{W_m \leq x, W_k \leq y\} = \Pr\{W_m \leq x\}$$

which depends only on x. Therefore (5.80) can be written

$$\mathrm{E}[N(x)N(y)] = \sum_{m=1}^{\infty} \sum_{k=1}^{m} \Pr\{W_m \leq x\} + \sum_{m=1}^{\infty} \sum_{k=m+1}^{\infty} \Pr\{W_m \leq x, W_k \leq y\} \tag{5.81}$$

5.8 The Expectation of the Extended Compound Point Process

The expectation of the extended compound point process is

$$\begin{aligned}
\mathrm{E}[X(t)] &= \mathrm{E}\left[\sum_{m=1}^{N(t)} Y(t, W_m)\right] \\
&= \mathrm{E}\left[\sum_{m=1}^{\infty} I(t, W_m)\right] \\
&= \sum_{m=1}^{\infty} \mathrm{E}[I(t, W_m)] \\
&= \sum_{m=1}^{\infty} \int_0^{\infty} \mathrm{E}[I(t, W_m)|W_m = x] d\Pr\{W_m \leq x\} \\
&= \sum_{m=1}^{\infty} \int_0^{\infty} \mathrm{E}[I(t, x)]\, d\Pr\{W_m \leq x\}
\end{aligned}$$

Assuming absolute convergence, the order of integration and summation may be interchanged so that

$$\begin{aligned}
\mathrm{E}[X(t)] &= \int_0^{\infty} \mathrm{E}[I(t, x)] d\sum_{m=1}^{\infty} \Pr\{W_m \leq x\} \\
&= \int_0^{t} \mathrm{E}[Y(t, x)] d\mathrm{E}[N(x)] \qquad (5.82)
\end{aligned}$$

Alternatively, this result could have been obtained by conditioning on $N(t)$. However, the above derivation is considerably simpler.

5.9 The Second Moment and the Variance of the Extended Compound Point Process

The second moment of the extended compound point process is

$$\mathrm{E}[X^2(t)] = \mathrm{E}\left[\left(\sum_{m=1}^{N(t)} Y(t, W_m)\right)^2\right] \qquad (5.83)$$

or

$$\begin{aligned}
\mathrm{E}[X^2(t)] &= \mathrm{E}\left[\left(\sum_{m=1}^{\infty} I(t, W_m)\right)^2\right] \\
&= \mathrm{E}\left[\sum_{m=1}^{\infty} I^2(t, W_m) + 2\sum_{m=1}^{\infty}\sum_{k=m+1}^{\infty} I(t, W_m) I(t, W_k)\right] \\
&= \sum_{m=1}^{\infty} \mathrm{E}[I^2(t, W_m)]
\end{aligned}$$

$$+2\sum_{m=1}^{\infty}\sum_{k=m+1}^{\infty} \mathrm{E}[I(t,W_m)I(t,W_k,)]$$
$$= \sum_{m=1}^{\infty}\int_0^{\infty} \mathrm{E}[I^2(t,x)]d\Pr\{W_m \le x\}$$
$$+2\sum_{m=1}^{\infty}\sum_{k=m+1}^{\infty}\int_0^{\infty}\int_0^{\infty} \mathrm{E}[I(t,x)]\mathrm{E}[I(t,y)]\,\partial_{xy}^2 \Pr\{W_m \le x, W_k \le y\} \tag{5.84}$$

Since for $y < x$ and $k \ge m+1$

$$\Pr\{W_m \le x, W_k \le y\} = \Pr\{W_k \le y\}$$

which depends only on y, we have for $y \in [0,x)$

$$\partial_{xy}^2 \Pr\{W_m \le x, W_k \le y\} = 0 \tag{5.85}$$

Thus, relation (5.84) can be written

$$\sum_{m=1}^{\infty}\int_0^{\infty} \mathrm{E}[I^2(t,x)]d\Pr\{W_m \le x\}$$
$$+2\sum_{m=1}^{\infty}\sum_{k=m+1}^{\infty}\int_0^{\infty}\int_x^{\infty} \mathrm{E}[I(t,x)]\mathrm{E}[I(t,y)]\,\partial_{xy}^2 \Pr\{W_m \le x, W_k \le y\}$$
$$= \int_0^{\infty} \mathrm{E}[I^2(t,x)]d\sum_{m=1}^{\infty} \Pr\{W_m \le x\}$$
$$+2\int_0^{\infty}\int_x^{\infty} \mathrm{E}[I(t,x)]\mathrm{E}[I(t,y)]\partial_{xy}^2 \sum_{m=1}^{\infty}\sum_{k=m+1}^{\infty} \Pr\{W_m \le x, W_k \le y\} \tag{5.86}$$

Note that

$$\Pr\{W_m \le x, W_k \le y\} = \Pr\{N(x) \ge m, N(y) \ge k\}$$

is true only when simultaneous arrivals cannot occur, i.e.

$$\Pr\{W_m \le x, W_k \le x\} = \Pr\{W_k \le x\} = \lim_{y\to x^-} \Pr\{W_m \le x, W_k \le y\}$$

Also, using (5.80)

$$\partial_{xy}^2 \mathrm{E}[N(x)N(y)] = \partial_{xy}^2 \sum_{m=1}^{\infty}\sum_{k=m+1}^{\infty} \Pr\{W_m \le x, W_k \le y\} \tag{5.87}$$

because

$$\partial_{xy}^2 \sum_{m=1}^{\infty}\sum_{k=1}^{m} \Pr\{W_m \le x\} = 0$$

Thus, using property (5.78) and relation (5.87), (5.86) can be written

$$\mathrm{E}[X^2(t)] = \int_0^t \mathrm{E}[Y^2(t,x)]d\mathrm{E}[N(x)] + 2\int_0^t \int_x^t \mathrm{E}[Y(t,x)]\mathrm{E}[Y(t,y)]\partial^2_{xy}\mathrm{E}[N(x)N(y)] \quad (5.88)$$

Moreover,

$$\begin{aligned}
\mathrm{E}^2[X(t)] &= \left(\int_0^t \mathrm{E}[Y(t,x)]d\mathrm{E}[N(x)]\right)^2 \\
&= \int_0^t \int_0^t \mathrm{E}[Y(t,x)]\mathrm{E}[Y(t,y)]d\mathrm{E}[N(y)]d\mathrm{E}[N(x)] \\
&= \int_0^t \int_0^x \mathrm{E}[Y(t,x)]\mathrm{E}[Y(t,y)]d\mathrm{E}[N(y)]d\mathrm{E}[N(x)] \\
&\quad + \int_0^t \int_x^t \mathrm{E}[Y(t,x)]\mathrm{E}[Y(t,y)]d\mathrm{E}[N(y)]d\mathrm{E}[N(x)] \\
&= \int_0^t \int_y^t \mathrm{E}[Y(t,x)]\mathrm{E}[Y(t,y)]d\mathrm{E}[N(y)]d\mathrm{E}[N(x)] \\
&\quad + \int_0^t \int_x^t \mathrm{E}[Y(t,x)]\mathrm{E}[Y(t,y)]d\mathrm{E}[N(y)]d\mathrm{E}[N(x)] \qquad (5.89)
\end{aligned}$$

Since the integrand in the second part of (5.89) is a symmetric function of x and y, (5.89) can be written

$$\mathrm{E}^2[X(t)] = 2\int_0^t \int_0^x \mathrm{E}[Y(t,x)]\mathrm{E}[Y(t,y)]d\mathrm{E}[N(y)]d\mathrm{E}[N(x)] \qquad (5.90)$$

Thus, combining (5.88) and (5.90) we obtain

$$\begin{aligned}
&\mathrm{Var}[X(t)] = \mathrm{E}[X^2(t)] - \mathrm{E}^2[X(t)] \\
&= \int_0^t \mathrm{E}[Y^2(t,x)]d\mathrm{E}[N(x)] + 2\int_0^t \int_x^t \mathrm{E}[Y(t,x)]\mathrm{E}[Y(t,y)]\partial^2_{xy}\mathrm{E}[N(x)N(y)] \\
&\quad -2\int_0^t \int_x^t \mathrm{E}[Y(t,x)]\mathrm{E}[Y(t,y)]\partial^2_{xy}\left(\mathrm{E}[N(y)]\mathrm{E}[N(x)]\right) \\
&= \int_0^t \mathrm{E}[Y^2(t,x)]d\mathrm{E}[N(x)] \\
&\quad +2\int_0^t \int_x^t \mathrm{E}[Y(t,x)]\mathrm{E}[Y(t,y)]\partial^2_{xy}\left(\mathrm{E}[N(x)N(y)] - \mathrm{E}[N(y)]\mathrm{E}[N(x)]\right) \\
&= \int_0^t \mathrm{E}[Y^2(t,x)]d\mathrm{E}[N(x)] + 2\int_0^t \int_x^t \mathrm{E}[Y(t,x)]\mathrm{E}[Y(t,y)]\partial^2_{xy}\mathrm{Cov}[N(x),N(y)] \quad (5.91)
\end{aligned}$$

This quasi-closed form result shows that it is possible to obtain $\mathrm{Var}[X(t)]$ in terms of the first two moments of $N(t)$ and $Y(t,x)$.

5.10 Higher Moments of the Extended Compound Point Process

Similarly we can obtain the r-th moment of the extended compound point process as follows

$$\mathrm{E}[X^r(t)] = \mathrm{E}\left[\left(\sum_{l=1}^{N(t)} Y(t,W_l)\right)^r\right]$$

$$= \mathrm{E}\left[\sum_{l_1=1}^{\infty}\dots\sum_{l_r=1}^{\infty} I(t,W_{l_1})\dots I(t,W_{l_r})\right]$$

$$= \sum_{l_1=1}^{\infty}\dots\sum_{l_r=1}^{\infty}\int_0^{\infty}\dots\int_0^{\infty} \mathrm{E}[I(t,x_1)\dots I(t,x_r)]\partial_{x_1\cdots x_r}\Pr\{W_{l_1}\le x_1,\dots,W_{l_r}\le x_r\}$$

$$= \sum_{l_1=1}^{\infty}\dots\sum_{l_r=1}^{\infty}\int_0^{\infty}\dots\int_0^{\infty} \mathrm{E}[I(t,W_{l_1})\dots I(t,W_{l_r})|W_{l_1}=x_1,\dots,W_{l_r}=x_r]$$
$$\partial_{x_1\cdots x_r}\Pr\{W_{l_1}\le x_1,\dots,W_{l_r}\le x_r\}$$

Assuming absolute convergence, and using the same procedure as in Section 5.4, we obtain

$$\mathrm{E}[X^r(t)] = \int_0^t\dots\int_0^t \mathrm{E}[Y(t,x_1)\dots Y(t,x_r)]\partial_{x_1\cdots x_r}\mathrm{E}[N(x_1)\cdots N(x_r)] \tag{5.92}$$

We can recover the first moment immediatelly. The second moment and the variance can also be recovered after careful substitution and if we take into account that for $0\le s\le t<\infty$

$$\mathrm{Cov}[X(s),X(t)] = \mathrm{Var}[X(s)] + \mathrm{Cov}[X(s),X(t)-X(s)]$$

(see Parzen[31]). The procedure, however, is more cumbersome and, thus, we preferred the previous way. Specific expressions for the higher moments will not be attempted here.

5.11 The Expectation and the Variance of the Number of Busy Channels in the $PP^X/G/\infty$ Queue

Infinite server queues have been used to model a broad variety of systems ranging from analyses of flows of populations to ecological studies of animals of given species in a variety of habitats. Historically, the first infinite server queue to be studied was the $M/M/\infty$ queue and its natural extension $M(t)/G/\infty$ –where $M(t)$ stands for a non-homogeneous Poisson process– (see Brown and Ross[5]). Similar systems have been developed for batch arrival processes (e.g. see Keilson and Seidman[26] and Chatterjee and Mukherjee[6]). Some results, also, exist for the $GI/G/\infty$ queue (see e.g. Puri[33] and Wolff[42]). Similar results

exist for infinite server queues with batch arrivals (see e.g. Liu et al.[27] and Liu and Templeton[28]). For classical approaches to problems related to queues with batch arrivals the reader is referred to Chaudhry and Templeton[7]. Here, following Kendall's notation, we consider the $PP^X/G/\infty$ queueing systems, in which batch arrivals occur as an arbitrary simple point process (PP). Service times are assumed to be mutually independent but may depend on the batch arrival time. They are also assumed to be independent of other arrival times and other service times. The size of the batch can, likewise, depend on its arrival time but it may not depend on other batches.

Formulas (5.82) and (5.91) can be used to derive the *expectation* and the *variance* of the number of busy channels in the infinite server $PP^X/G/\infty$ queue with random batch arrivals. The $PP^X/G/\infty$ queue can be considered as a special case of the extended compound point process in which

$$Y(t,x) = \sum_{j=1}^{B} I(S_j, t-x)$$

where,

1) B is a discrete random variable that represents the size of a batch.

2) S_j is a sequence of nonnegative independently and identically distributed random variables, with distribution function $F(\cdot;x)$ and complementary distribution function $F^c(\cdot;x)$, representing the service time of the j-th unit of a batch that arrived at time x. In what follows we assume that the first two moments of the service times are finite.

3) $I(\cdot)$ is an indicator function defined as follows

$$I(S_j, t-x) = \begin{cases} 1, & 0 \le t-x < S_j \\ 0, & \text{otherwise} \end{cases}$$

$Y(t,x)$ is a simple compound process. For notational simplicity, let for a batch arriving at time $W_i = x$

$$\mathrm{E}[B|W_i = x] = \mu_1(x)$$

and

$$\mathrm{E}[B(B-1)|W_i = x] = \mu_2(x)$$

Then

$$\mathrm{E}[Y(t,x)] \quad = \quad \mathrm{E}[B|W_i = x]\mathrm{E}[I(S_j, t-x)]$$

$$
\begin{aligned}
&= \mathrm{E}[B|W_i = x] \Pr\{S_j > t - x\} \\
&= \mathrm{E}[B|W_i = x][1 - F(t - x; x)] \\
&= \mu_1(x) F^c(t - x; x) \qquad (5.93)
\end{aligned}
$$

Substituting (5.93) into (5.82) we obtain

$$
\mathrm{E}[X(t)] = \int_0^t \mu_1(x) F^c(t - x; x) d\mathrm{E}[N(x)] \qquad (5.94)
$$

Note that for $\mathrm{E}[X(t)]$ the independence assumption of the sequence of $Y(t, W_j)$ is not required. Also, since we assumed that service times are mutually independent, the number of units of a batch arriving at time x that are still in service at time $t, (t > x)$, conditional on the batch size $B = n$, is binomially distributed with parameters n and p, where $p = F^c(t-x; x)$. Therefore,

$$
\mathrm{E}[Y^2(t, x)|B = n] = n(n-1)p^2 + np
$$

From the law of total probability

$$
\begin{aligned}
\mathrm{E}[Y^2(t, x)] &= p^2 \mathrm{E}[B(B-1)|W_i = x] + p\mathrm{E}[B|W_i = x] \\
&= [F^c(t - x; x)]^2 \mu_2(x) + F^c(t - x; x)\mu_1(x) \qquad (5.95)
\end{aligned}
$$

Thus, (5.91) becomes

$$
\begin{aligned}
\mathrm{Var}[X(t)] = &\int_0^t \mu_1(x) F^c(t - x; x) d\mathrm{E}[N(x)] + \int_0^t \mu_2(x)[F^c(t - x; x)]^2 d\mathrm{E}[N(x)] \\
&+2 \int_0^t \int_x^t \mu_1(x) F^c(t - x; x) \mu_1(y) F^c(t - y; y) \partial^2_{xy} \mathrm{Cov}[N(x), N(y)] \qquad (5.96)
\end{aligned}
$$

We consider now two important special cases.

Special Case 1. *The $M(t)^X/G/\infty$ Queue*

First, for point processes with independent increments we recover the $M(t)^X/G/\infty$ queue. Thus, if $\lambda(t)$ is the arrival rate, we obtain

$$
\mathrm{E}[X(t)] = \int_0^t \mu_1(x) F^c(t - x; x) \lambda(x) dx \qquad (5.97)
$$

Also

$$
\partial^2_{xy} \mathrm{Cov}[N(x), N(y)] = \partial^2_{xy} \mathrm{Cov}[N(x), N(y) - N(x)] + \partial^2_{xy} \mathrm{Var}[N(x)] = 0
$$

Thus

$$\begin{aligned}\mathrm{Var}[X(t)] &= \int_0^t \mu_1(x)F^c(t-x;x)\lambda(x)dx + \int_0^t \mu_2(x)[F^c(t-x;x)]^2\lambda(x)dx \\ &= \int_0^t [\mu_1(x) + \mu_2(x)F^c(t-x;x)]F^c(t-x;x)\lambda(x)dx \qquad (5.98)\end{aligned}$$

Special Case 2. *The $GI^X/G/\infty$ Queue*

Next, if the arrival process is a renewal counting process with renewal density $m(\cdot)$, then the system is the $GI^X/G/\infty$ queue. In this case we recover the well known result for the expectation:

$$\mathrm{E}[X(t)] = \int_0^t \mu_1(x)F^c(t-x;x)m(x)dx \qquad (5.99)$$

Using a result for the differential form of $\mathrm{E}[N(x)N(y)]$ (see Daley and Vere-Jones[10]) an expression for the variance is obtained as

$$\begin{aligned}\mathrm{Var}[X(t)] = &\int_0^t \mu_1(x)F^c(t-x;x)m(x)dx + \int_0^t \mu_2(x)\left[F^c(t-x;x)\right]^2 m(x)dx \\ &+2\int_0^t\int_x^t \mu_1(x)F^c(t-x;x)\mu_1(y)F^c(t-y;y)m(x)[m(y-x)-m(y)]dydx \qquad (5.100)\end{aligned}$$

Although some results for the number of busy channels of the $GI^X/G/\infty$ queue exist in the literature (e.g. see Liu et al.[27]), an expression for the variance does not seem to have appeared so far. Thus, expression (5.100) appears to be a novel one. Moreover, existing work on the $GI^X/G/\infty$ has restricted the service times to be identically distributed over time, while our model allows for variation of the service time distribution over time.

Additionally, assuming that $\mu_1(x) = \mu_1$, $F^c(t-x;x) = F^c(t-x)$, and mean inter-arrival time ν, we can recover the well-known result for the steady state expected number of busy channels of the $GI^X/G/\infty$ queue:

$$\lim_{t\to\infty} \mathrm{E}[X(t)] = \frac{\mu_1 \mathrm{E}[S]}{\nu} \qquad (5.101)$$

5.12 Other Applications

5.12.1 The Expectation and the Variance of the Total Backlog of the $PP/G/\infty$ Queue

The *total backlog* in an infinite server queue is the total remaining service time still to be provided to the units currently in the system. For a $PP/G/\infty$ queue in which arrivals occur

as an arbitrary simple point process (PP), it can be described by an extended compound point process -using the same assumptions and notations as previously- in which

$$Y(t,x) = \begin{cases} S-(t-x) & \text{for } 0 \leq t-x < S \\ 0 & \text{for } S \leq t-x < \infty \end{cases}$$

Thus

$$\mathrm{E}[Y(t,x)] = \int_{t-x}^{\infty} (y-t+x)\partial_y F(y;x)$$

and

$$\mathrm{E}[Y^2(t,x)] = \int_{t-x}^{\infty} (y-t+x)^2\partial_y F(y;x)$$

Therefore

$$\mathrm{E}[X(t)] = \int_0^t \int_{t-x}^{\infty} (y-t+x)\partial_y F(y;x) d\mathrm{E}[N(x)] \tag{5.102}$$

and

$$\begin{aligned} \mathrm{Var}[X(t)] &= \int_0^t \int_{t-x}^{\infty} (w-t+x)^2 \partial_w F(w;x) d\mathrm{E}[N(x)] \\ &+2\int_0^t \int_x^t \left(\int_{t-x}^{\infty} (u-t+x)\partial_u F(u;x) \int_{t-y}^{\infty} (v-t+y)\partial_v F(v;x)\right) \partial^2_{xy}\mathrm{Cov}[N(x),N(y)] \end{aligned} \tag{5.103}$$

<u>Special Case 1.</u> *The $M(t)/G/\infty$ Queue*

$$\mathrm{E}[X(t)] = \int_0^t \int_{t-x}^{\infty} (y-t+x)\partial_y F(y;x)\lambda(x)dx \tag{5.104}$$

and

$$\mathrm{Var}[X(t)] = \int_0^t \int_{t-x}^{\infty} (w-t+x)^2\partial_w F(w;x)\lambda(x)dx \tag{5.105}$$

<u>Special Case 2.</u> *The $GI/G/\infty$ Queue*

$$\mathrm{E}[X(t)] = \int_0^t \int_{t-x}^{\infty} (y-t+x)\partial_y F(y;x)m(x)dx \tag{5.106}$$

and

$$\begin{aligned} \mathrm{Var}[X(t)] &= \int_0^t \int_{t-x}^{\infty} (w-t+x)^2\partial_w F(w;x)m(x)dx \\ &+2\int_0^t \int_x^t \left(\int_{t-x}^{\infty} (u-t+x)\partial_u F(u;x) \int_{t-y}^{\infty} (v-t+y)\partial_v F(v;x)\right) \\ &\qquad m(x)[m(y-x)-m(y)]dydx \end{aligned} \tag{5.107}$$

Additionally assuming that $F^c(t-x;x) = F^c(t-x)$ and mean inter-arrival time ν we can obtain the result for the steady state total expected backlog of the $GI/G/\infty$ queue:

$$\lim_{t\to\infty} \mathrm{E}[X(t)] = \frac{\mathrm{E}[S^2]}{2\nu} \tag{5.108}$$

5.12.2 The Expectation and the Variance of the Total Revenue from Long Distance Phone Calls

The revenue from a given long distance phone call is, usually, a function of the duration of the conversation. Here we assume that for a conversation starting at time x, the revenue from that phone call at time t is

$$r(t,x,S) = \begin{cases} \alpha(t-x) & \text{for } 0 \le t-x < S \\ \alpha S & \text{for } S \le t-x < \infty \end{cases}$$

where S is a nonnegative random variable which represents the duration of a conversation starting at time x, with distribution function $F(\cdot;x)$ and complementary distribution function $F^c(\cdot;x)$. Then

$$\begin{aligned} \mathrm{E}[r(t,x,S)] &= \int_0^\infty r(t,x,y)\partial_y F(y;x) \\ &= \int_0^{t-x} \alpha y \partial_y F(y;x) + \int_{t-x}^\infty \alpha(t-x)\partial_y F(y;x) \\ &= \alpha \int_0^{t-x} y\partial_y F(y;x) + \alpha(t-x)F^c(t-x;x) \end{aligned}$$

and

$$\begin{aligned} \mathrm{E}[r^2(t,x,S)] &= \int_0^\infty r^2(t,x,y)\partial_y F(y;x) \\ &= \int_0^{t-x} (\alpha y)^2 \partial_y F(y;x) + \alpha^2(t-x)^2 F^c(t-x;x) \end{aligned} \tag{5.109}$$

Assuming that the arrival process of long distance phone calls is an arbitrary simple point process, then the total revenue $X(t)$ over $[0,t]$ is an extended compound point process in which

$$X(t) = \sum_{j=1}^{N(t)} r(t, W_j, S_j)$$

where the random variable S_j represents the duration of the j-th conversation, assumed independently and identically distributed. Thus, the total expected revenue at time t is

$$\mathrm{E}[X(t)] = \int_0^t \mathrm{E}[r(t,x,S)]d\mathrm{E}[N(x)]$$

$$= \int_0^t \left\{ \alpha \int_0^{t-x} y \partial_y F(y;x) + \alpha(t-x) F^c(t-x;x) \right\} d\mathrm{E}[N(x)] \quad (5.110)$$

and

$$\begin{aligned} \mathrm{Var}[X(t)] &= \alpha^2 \int_0^t \left\{ \int_0^{t-x} y^2 \partial_y F(y;x) + (t-x)^2 F^c(t-x;x) \right\} d\mathrm{E}[N(x)] \\ &+2 \int_0^t \int_x^t \mathrm{E}[r(t,x,S)]\mathrm{E}[r(t,y,S)] \partial^2_{xy} \mathrm{Cov}[N(x), N(y)] \quad (5.111) \end{aligned}$$

In general, the expected revenue will grow infinitelly large as $t \to \infty$ and, thus, we do not investigate the limiting case. It may be interesting, however, to investigate another measure, the *revenue rate* defined as follows

$$\tilde{r}(t) = \frac{d}{dt}\mathrm{E}[X(t)]$$

Assume for simplicity that

$$d\mathrm{E}[N(t)] = m(t)dt$$

Then

$$\begin{aligned} \frac{d}{dt}\mathrm{E}[X(t)] &= \frac{d}{dt} \int_0^t \left\{ \alpha \int_0^{t-x} y \partial_y F(y;x) + \alpha(t-x) F^c(t-x;x) \right\} d\mathrm{E}[N(x)] \\ &= \int_0^t \alpha \frac{d}{dt} \left\{ \int_0^{t-x} y \partial_y F(y;x) + (t-x) F^c(t-x;x) \right\} d\mathrm{E}[(x)] \\ &+\alpha \left[\int_0^{t-x} y \partial_y F(y;x) + (t-x) F^c(t-x;x) \right]_{x=t} m(t) \\ &= \alpha \int_0^t m(t)[(t-x) - (t-x)] \partial_y F(t-x;x) + \alpha \int_0^t m(x) F^c(t-x;x) dx \\ &= \alpha \int_0^t m(x) F^c(t-x;x) dx \quad (5.112) \end{aligned}$$

Additionally assuming that $F^c(t-x;x) = F^c(t-x)$ and that $\lim_{t\to\infty} m(t) = 1/\nu$, where ν is the mean inter-arrival time, we can recover the well-known result for the steady state expected revenue rate:

$$\tilde{r} = \lim_{t\to\infty} \tilde{r}(t) = \frac{\alpha \mathrm{E}[S]}{\nu} \quad (5.113)$$

5.12.3 A Special Model: Moments of the Underlying Point Process Determined Empirically

To the extent that one is only interested in computing the first two moments of the extended compound point process, available techniques to describe the two moments of the underlying point process fail to encompass situations which are not amenable to modeling

as a renewal process or as a non-homogeneous Poisson process. We provide here an example that demonstrates the utility of our model in such situations. In what follows we assume that we have *empirically* determined the time dependent forms for the first two moments of the underlying point process which cannot be considered as generated from either a renewal process or a non-homogeneous Poisson process.

Let us assume that the expectation of the underlying point process has the empirically determined form:

$$\mathrm{E}[N(t)] = a(1 - e^{-\beta t}) + b(t + \sin \omega t) \tag{5.114}$$

where $a, b, \beta, \omega > 0$ and $t \geq 0$. This form guarantees that $\mathrm{E}[N(0)] = 0$, Moreover, $\mathrm{E}[N(t)]$ is a non-decreasing function that oscillates asymptotically about the line $a + bt$ with period $2\pi/\omega$. Clearly then

$$d\mathrm{E}[N(t)] = [a\beta e^{-\beta t} + b(1 + \omega \cos \omega t)]dt \tag{5.115}$$

A function of this form would be inappropriate to describe the renewal density function of a renewal process since it does not converge as $t \to \infty$. It would be appropriate for a non-homogeneous Poisson process. However, the non-homogeneous Poisson process requires that the covariance of the increments be zero. To demonstrate the generality of our model we can assume that the covariance function of the increments is, in general, nonzero, say of the following empirically determined form:

$$\mathrm{Cov}[N(t) - N(s), N(s)] = c(1 - e^{-\gamma s})(1 - e^{-\gamma(t-s)})e^{-\gamma(t-s)} \tag{5.116}$$

where $\gamma > 0$ and $0 < s \leq t$. This form guarantees that for $s = t$, $\mathrm{Cov}[N(t) - N(s), N(s)] = 0$ and again that for $(t - s) \to \infty$, $\mathrm{Cov}[N(t) - N(s), N(s)] = 0$.

We obtain then

$$\begin{aligned} \partial_{xy}\mathrm{Cov}[N(y) - N(x), N(x)] &= \partial_{xy}\mathrm{Cov}[N(y), N(x)] \\ &= [4c\gamma^2 e^{-2\gamma(y-x)} - c\gamma^2 e^{-\gamma(y-x)} - 2c\gamma^2 e^{-\gamma(2y-x)}]dydx \end{aligned} \tag{5.117}$$

Substituting (5.115) and (5.117) into (5.82) and (5.91), we obtain

$$\mathrm{E}[X(t)] = \int_0^t \mathrm{E}[Y(t, x)][a\beta e^{-\beta x} + b(1 + \omega \cos \omega x)]dx \tag{5.118}$$

and

$$\mathrm{Var}[X(t)] = \int_0^t \mathrm{E}[Y^2(t,x)][a\beta e^{-\beta x} + b(1+\omega\cos\omega x)]dx$$
$$+2\int_0^t\int_x^t \mathrm{E}[Y(t,x)]\mathrm{E}[Y(t,y)](4c\gamma^2 e^{-2\gamma(y-x)} - c\gamma^2 e^{-\gamma(y-x)} - 2c\gamma^2 e^{-\gamma(2y-x)})dydx \quad (5.119)$$

Special Case 1. *The number of busy channels in a $PP/G/\infty$ queue*

For example consider a $PP/G/\infty$ queue in which the arrival rate of the customers is time dependent and is dictated by expression (5.115). Assume for simplicity that the complementary distribution function of the service times $F^c(t-x;x) = F^c(t-x)$. Let $X(t)$ be the number of busy channels. Using (5.115) in (5.94) with $\mu_1(x) = 1$, we have

$$\mathrm{E}[X(t)] = \int_0^t F^c(t-x)[a\beta e^{-\beta x} + b(1+\omega\cos\omega x)]dx$$
$$= a\beta\int_0^t F^c(t-x)e^{-\beta x}dx + b\int_0^t F^c(t-x)dx + b\omega\int_0^t F^c(t-x)\cos\omega x dx \quad (5.120)$$

Taking the limit as $t\to\infty$,

$$\lim_{t\to\infty}\mathrm{E}[X(t)] = b\mathrm{E}[S] + b\omega\lim_{t\to\infty}\int_0^t F^c(t-x)\cos\omega x dx \quad (5.121)$$

which is a periodic function that oscillates about the constant value $b\mathrm{E}[S]$.

Similarly,

$$\mathrm{Var}[X(t)] = \int_0^t F^c(t-x)][a\beta e^{-\beta x} + b(1+\omega\cos\omega x)]dx$$
$$+2\int_0^t\int_x^t F^c(t-x)F^c(t-y)(4c\gamma^2 e^{-2\gamma(y-x)} - c\gamma^2 e^{-\gamma(y-x)} - 2c\gamma^2 e^{-\gamma(2y-x)})dydx \quad (5.122)$$

and, taking the limit as $t\to\infty$, we obtain

$$\lim_{t\to\infty}\mathrm{Var}[X(t)] =$$
$$= b\mathrm{E}[S] + b\omega\lim_{t\to\infty}\int_0^t F^c(t-x)\cos\omega x dx$$
$$+\lim_{t\to\infty}2\int_0^t\int_x^t F^c(t-x)F^c(t-y)(4c\gamma^2 e^{-2\gamma(y-x)} - c\gamma^2 e^{-\gamma(y-x)} - 2c\gamma^2 e^{-\gamma(2y-x)})dydx$$
$$(5.123)$$

Special Case 2. *The number of busy channels in a $PP/M/\infty$ queue*

Assume again that the arrival rate is time dependent and is dicteted by (5.115). Further assume that the service times are negative exponentially distributed with parameter μ, so that

$$F^c(t-x) = e^{-\mu(t-x)} \tag{5.124}$$

Substituting (5.124) into (5.120) and (5.122), we obtain

$$\mathrm{E}[X(t)] = \frac{a\beta}{\mu}e^{-\beta t}(1-e^{-\mu t}) + \frac{b}{\mu}(1-e^{-\mu t}) + \frac{b\omega}{\mu^2+\omega^2}(\mu\cos\omega t + \omega\sin\omega t - \mu e^{-\mu t}) \tag{5.125}$$

and

$$\begin{aligned}\mathrm{Var}[X(t)] &= \frac{a\beta}{\mu}e^{-\beta t}(1-e^{-\mu t}) + \frac{b}{\mu}(1-e^{-\mu t}) + \frac{b\omega}{\mu^2+\omega^2}(\mu\cos\omega t + \omega\sin\omega t - \mu e^{-\mu t})\\ &+8c\gamma^2\left[\frac{1-e^{-(\mu+2\gamma)t}}{(\mu-2\gamma)(\mu+2\gamma)} - \frac{1-e^{-2\mu t}}{(\mu-2\gamma)(2\mu)}\right] - 2c\gamma^2\left[\frac{1-e^{-(\mu+\gamma)t}}{(\mu-\gamma)(\mu+\gamma)} - \frac{1-e^{-2\mu t}}{(\mu-\gamma)(2\mu)}\right]\\ &-4c\gamma^2\left[\frac{e^{-\gamma t}-e^{-(\mu+2\gamma)t}}{(\mu-2\gamma)(\mu+\gamma)} - \frac{e^{-\gamma t}-e^{-2\mu t}}{(\mu-2\gamma)(2\mu-\gamma)}\right]\end{aligned} \tag{5.126}$$

Taking the limit as $t\to\infty$ we obtain

$$\lim_{t\to\infty}\mathrm{E}[X(t)] = \frac{b}{\mu} + \lim_{t\to\infty}\frac{b\omega}{\mu^2+\omega^2}(\mu\cos\omega t + \omega\sin\omega t) \tag{5.127}$$

and

$$\begin{aligned}\lim_{t\to\infty}\mathrm{Var}[X(t)] &= \frac{b}{\mu} + \lim_{t\to\infty}\frac{b\omega}{\mu^2+\omega^2}(\mu\cos\omega t + \omega\sin\omega t)\\ &+\frac{8c}{(\mu-2\gamma)(\mu+2\gamma)} - \frac{8c\gamma^2}{(\mu-2\gamma)(2\mu)} - \frac{2c\gamma^2}{(\mu-\gamma)(\mu+\gamma)} + \frac{2c\gamma^2}{(\mu-\gamma)(2\mu)}\\ &= \frac{b}{\mu} + \lim_{t\to\infty}\frac{b\omega}{\mu^2+\omega^2}(\mu\cos\omega t + \omega\sin\omega t) + c\gamma^2\frac{3\mu+2\gamma}{\mu(\mu+2\gamma)(\mu+\gamma)}\end{aligned} \tag{5.128}$$

Let

$$\phi = \tan^{-1}\frac{\mu}{\omega}$$

Then the periodic terms in expressions (5.125), (5.126), (5.127) and (5.128) can be written as

$$\frac{b\omega}{\sqrt{\mu^2+\omega^2}}[\sin\phi\cos\omega t + \cos\phi\sin\omega t] = \frac{b\omega}{\sqrt{\mu^2+\omega^2}}\sin(\omega t+\phi)$$

Therefore, the expectation and the variance of the number of busy channels have a difference of phase ϕ with the expectation of the arrival point process, which is the input to our system. Note that the parameters b, c and γ must satisfy the condition

$$\frac{b}{\mu} - \frac{b\omega}{\sqrt{\mu^2 + \omega^2}} + c\gamma^2 \frac{3\mu + 2\gamma}{\mu(\mu + 2\gamma)(\mu + \gamma)} > 0$$

for all $\mu > 0$ so that the variance does not take negative values.

5.13 Conclusion

In this chapter, motivated by the filtered Poisson process, we extended the renewal process and presented the notions of the filtered renwal process and the extended compound renewal process. We derived a second type Volterra integral equation for the characteristic function of the extended compound renewal process and we investigated special cases in which the integral equation can be solved. From the same integral equation we derived the first moment of the extended compound renewal process and recursive relationship for the higher order moments. We demonstrated how one can use the extended compound renewal process to model infinite server queues with bulk arrivals and obtain results for measures like the number of busy channels, the output rate, the total backlog and the revenue rate. We carried our analysis one step further and showed that similar results can be obtained for a more general class of stochastic process the extended compound point process. Finally, we demonstated how the extended compound point process can be utilized to model queueing systems with arrivals occuring according to a point process that cannot be described as either a renewal counting process or a non-homogeneous Poisson process (see Gakis and Sivazlian[16, 17]).

CHAPTER 6

APPLICATIONS AND EXTENSIONS FOR FUTURE RESEARCH

6.1 Introduction

In this chapter we present some ideas that we feel form the basis for applications of the results obtained in this work and extensions for future research.

6.2 A Continuous Review Inventory Model

6.2.1 Description of the Policy and Notation

We consider an (s, S) continuous review inventory problem with constant positive lead time. The inventory replenishment policy is thus: whenever the inventory level falls below $s > 0$, a quantity is ordered to bring the inventory level to a positive value $S > s$. In general, the quantity ordered is not constant and is dictated by the demand pattern. It is always greater than $S - s$. The order is replenished after l time units.

The demand for the product has the following structure. The time intervals between customer arrivals triggering withdrwals from inventory form a sequence of continuous random variables $\{Y_i\}$ with known finite expectation E$[Y]$. The amount of withdrawal from inventory to satisfy for each customer's demand form a sequence of continuous random variables $\{T_i\}$ with known finite expectation E$[T]$. If the $\{Y_i\}$'s are identically and independently distributed they form an ordinary renewal process. Similarly, the $\{T_i\}$'s are identically and independently distributed. Likewise, the $\{Ti\}$'s are independently distributed from the $\{Y_i\}$'s. Any unsatisfied orders are backlogged.

Our objective is to describe an approach to the problem that will lead to the determination of the average time interval elapsed between two successive replenishment orders and the average stock and backlog levels.

We shall use these three statistics to formulate an expected cost function per unit time, in the steady state, expressed in terms of the decision variables s and S.

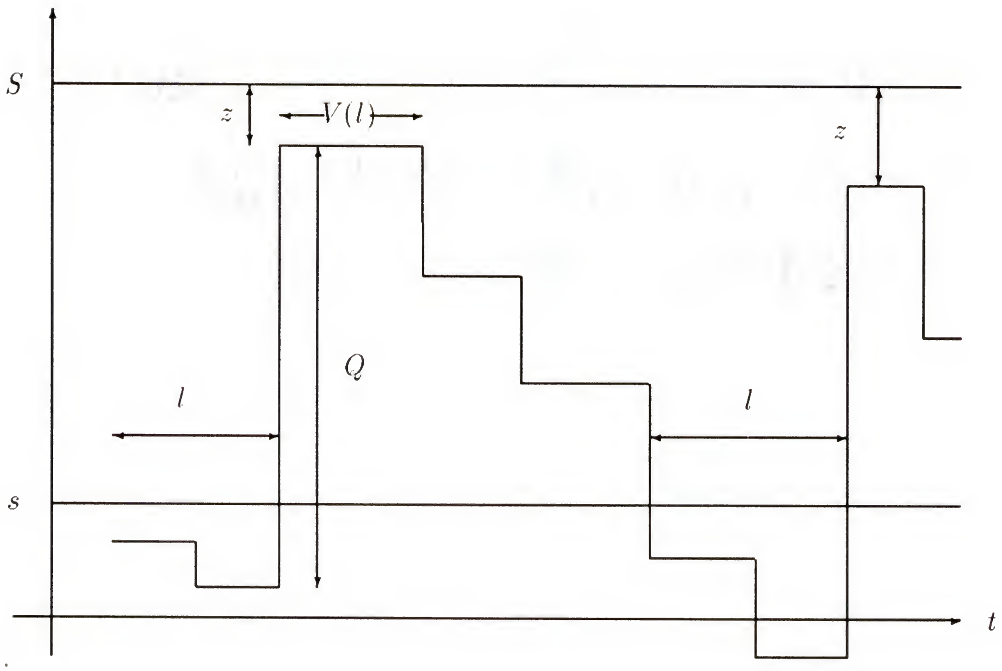

Figure 6.1. A continuous review inventory model

6.2.2 The Model

Consider a cycle between two successive order replenishments (see Figure 6.1). Let

- $N(x)$ = the number of customers which have arrived from the beginning of a cycle when a total of x units of demand have been generated,
- $N^0(t)$ = the number of customers which have arrived from the beginning of a cycle when a total of t units of time have been elapsed,
- $M_T(x) =$ E[N(x)],
- $m_T(x) = \frac{d}{dt} M_T(x)$,
- X = the cycle length,
- X^0 the time elapsed from the beginning of a cycle till an order is placed,

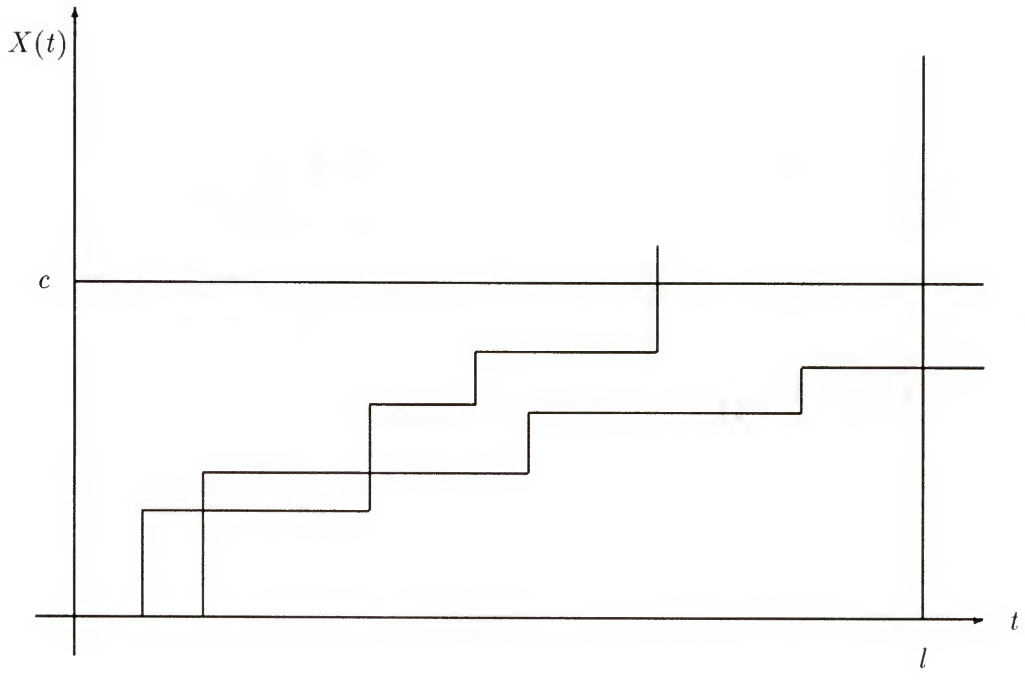

Figure 6.2. Realizations of the compound renewal process

- X^+ = the time elapsed from the moment a order was placed till either the order was replenished or the stock fell below zero,
- X^- = the time elapsed from the moment the inventory fell below zero till an order was replenished,
- $V(x)$ = the forward recurrence "time" of the counting process formed by the sequence $\{Y_i\}$.

Then

$$X = X^0 + X^+ + X^- = X^0 + l \tag{6.1}$$

and

$$X^0 = Y_1 + Y_2 + \cdots + Y_{N(S-s-z)+1}$$

The expectation of X^0 can be easily determined.

Note that in general $S - s - z$ may take negative values, i.e. immediatelly after the replenishment of an order the inventory level may be below s. In this case we assume that an order is placed immediately.

Note that

$$\mathrm{E}[X] = \mathrm{E}[X^0] + l \tag{6.2}$$

The area behind the inventory curve is then

$$A(S - s - z) = T_1 Y_1 + (T_1 + T_2) Y_3 + \cdots + (T_1 + T_2 + \cdots T_{N(S-s-z)}) Y_{N(S-s-z)+1}$$

Since

$$W_i = T_1 + T_2 + \cdots + T_i \quad \text{for } i = 1, 2, \ldots$$

Then

$$A(S - s - z) = W_1 Y_1 + W_2 Y_3 + \cdots + W_{N(S-s-z)} Y_{N(S-s-z)+1}$$

$Z(S - s - z)$ is a filtered renewal process in which the response function is

$$w_i(t, W_i, Y_{i+1}) = W_i Y_{i+1}$$

and it is easy to determine its expectation.

In order to determine the lengths X^+ and X^- and the areas B^+ and B^- we need to study more closely the behavior of the filtered renewal process. The following general problem relates directly to our problem:

Given a compound renewal process $X(t)$:

$$X(t) = T_1 + T_2 + \cdots + T_N^0(t)$$

and two barriers one on the t-axis denoted by l and one on the $X(t)$-axis denoted by c we want to find

1. $X^+ = \min[l, \min\{t : X(t) > c\}]$
2. $\mathrm{E}[X(X^+)]$
3. $B^+(X^+) = W_1 T_1 + W_2 T_2 + \cdots + W_{N^0(X^+)} T_{N^0(X^+)}$

4. $B^-(X^+) = 0$ if $X^+ = l$ or $B^-(X^+) = (l - X^+)V(c) + [l - W_{N(X^+)+1}]T_{N(X^+)+1} + [l - W_N^0(l)]T_{N^0(l)}$ if $X^+ < l$.

(see Figure 6.2)

In the context of our inventory model $c = S - s - V(S - z - s)$.

6.3 Infinite Server Queues in Tandem

A natural extension of the infinite server queue that we have considered in Chapter 5 is a system of several infinite server queues in tandem.

6.3.1 System Description

Consider a service system where customers are served through a set of $M(M \geq 1)$ service stations. Each service station has an infinite number of servers. The service time in each station is a non-negative random variable $S_i, (i = 1, 2, \ldots, M)$. Arrivals occur according to an arbitrary point process $\{N(t), t \geq t\}$ (possibly in batches). The output of one station is the input to the subsequent station. Concerning the sequence of the jobs we can consider the following cases:

- service can be rendered in any order; in this case we can investigate the problem of optimal sequencing;
- service has to be rendered in a certain order.

Concerning the properties of service time distributions we may consider cases such as

- service times are mutually independently distributed and time homogeneous;
- service times depend on the service times of previous stations;
- service time distributions are time dependent or non-homogeneous, i.e. depend on the time service is rendered.

6.3.2 Related Problems

Let $X_i(t), i = 1, 2, \ldots, M$ the number of busy channels of station i at time t. Then

$$X(t) = X_1(t) + X_2(t) + \cdots + X_M(t)$$

is the total number of busy channels in the system. Likewise let $\tilde{X}_i(t), i = 1, 2, \ldots, M$ be the total backlog of the customers currently in station i at time t. Then

$$\tilde{X}(t) = \tilde{X}_1(t) + \tilde{X}_2(t) + \cdots + \tilde{X}_M(t)$$

is the total backlog of the system. One can easily identify four important measures of performance:

- the expected number of busy channels in station i $(i = 1, 2, \ldots, M)$ $\mathrm{E}[X_i(t)]$;
- the total expected number of busy channels $\mathrm{E}[X(t)]$;
- the total expected backlog of station i $(i = 1, 2, \ldots, M)$ $\mathrm{E}[\tilde{X}_i(t)]$;
- the total expected backlog of the system $E[\tilde{X}(t)]$.

Likewise, one may be interested in determining higher order moments of the above mentioned random variables or their distributions. For the case of mutually independent and time homogeneous service times preliminary research shows that the total number of busy channels and the total system backlog are sequence invariant. When, however, service times depend on the sequence or are time non-homogeneous the investigation of optimal sequencing properties are appropriate.

6.4 Joint Characteristic Functions

We have already discussed the methodologies that have been utilized to obtain results in renewal theory. One of the methods that have not been used yet is that of working via the joint characteristic function of the renewal process, i.e.

$$\mathrm{E}[e^{\mathrm{i}s_1 N(t_1) + \mathrm{i}s_2 N(t_2)}]$$

Although in this work we have obtained all the information contained into the characteristic function of the renewal counting process its investigation may provide us with valuable insight for the problem of determination of the joint characteristic function of the extended compound renewal process (see discussion in Section 5.1) which presently seems a difficult task to handle.

6.5 Asymptotic Normality Properties

The existence of Central Limit Theorems for renewal counting processes has been well-known. Roginsky[34] has recently worked on Central Limit Theorems for renewal and cumulative processes, complemented existing results and improved error terms. Now that joint distributions of the number of renewals and of of order statistics of waiting times are available, a natural next step would be to investigate asymptotic normality properties for them. Specifically, the following distributions may asymptotically approach normality for integers $n_1 \leq n_2$ and real numbers $0 < t_1 < t_2 < t_3 < \infty$

1.

$$\Pr\{N(t_1) = n_1, N(t_2) = n_2\}$$

as both $t_1 \rightarrow \infty$ and $t_2 \rightarrow \infty$

2.

$$\Pr\{N(t_2) - N(t_1) = n_1\}$$

as $t_2 - t_1 \rightarrow \infty$

3.

$$\Pr\{N(t_2) - N(t_1) = n_1, N(t_3) - N(t_2)\}$$

as both $t_2 - t_1 \rightarrow \infty$ and $t_3 - t_2 \rightarrow \infty$

The investigation of normality properties may provide useful approximations for many applications.

6.6 The Concept of the Parent Distribution

Definition. In this section we investigate the problem of determining the distribution– from now on referred to as the *parent distribution*– and the corresponding random variable Z as the *parent random variable*- that generates order statistics distributed identically to the order statistics generated by the waiting times of a point process. But first we derive a general result that allows us to determine the parent distribution (assumed to have a probability density function) from an arbitrary distribution of order statistics.

<u>Lemma 1</u> *Let $F_Z(x)$ denote the distribution function of the parent distribution to be determined. Let $F_{Y_i}(x), i = 1, 2, \ldots n$ denote the known marginal distribution functions of the order statistics. Then*

$$F_Z(x) = \frac{\sum_{i=1}^{n} F_{Y_i}(x)}{n} \tag{6.3}$$

<u>Proof.</u> We know (see Mood et al.[30]):

$$f_{Y_i}(x) = \frac{n!}{(i-1)!(n-i)!}[F_Z(x)]^{i-1}[1 - F_Z(x)]^{n-i} f_Z(x) \tag{6.4}$$

If we sum over i we have

$$\begin{aligned}
\sum_{i=1}^{n} f_{Y_i}(x) &= \sum_{i=1}^{n} \frac{n!}{(i-1)!(n-i)!}[F_Z(x)]^{i-1}[1 - F_Z(x)]^{n-i} f_Z(x) \\
&= n \sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!}[F_Z(x)]^{j}[1 - F_Z(x)]^{n-1-j} f_Z(x) \\
&= n[F_Z(x) + 1 - F_Z(x)]^{n-1} f_Z(x) \\
&= n f_Z(x)
\end{aligned}$$

It follows immediately for $x \leq t$ that

$$f_Z(x) = \frac{\sum_{i=1}^{n} f_{Y_i}(x)}{n} \tag{6.5}$$

and

$$F_Z(x) = \frac{\sum_{i=1}^{n} F_{Y_i}(x)}{n} \tag{6.6}$$

Using (6.6) for the order statistics of a renewal process we obtain for $x \leq t$

$$\begin{aligned}
F_Z(x) &= \frac{1}{n} \sum_{i=1}^{n} \Pr\{W_i \leq x | N(t) = n\} \\
&= \frac{1}{n} \sum_{i=1}^{n} \Pr\{N(x) \geq i | N(t) = n\} \\
&= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=i}^{n} \Pr\{N(x) = j | N(t) = n\} \\
&= \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{j} \Pr\{N(x) = j | N(t) = n\} \\
&= \frac{1}{n} \sum_{j=1}^{n} j \Pr\{N(x) = j | N(t) = n\} \\
&= \frac{\mathrm{E}[N(x) | N(t) = n]}{n}
\end{aligned} \tag{6.7}$$

and, of course, for $x > t$,

$$F_Z(x) = 1$$

An intuitive explanation to this is that the proportion of renewals up to time x given the number of renewals up to time $t \geq x$ in an infinitely large number of sample realizations of the renewal process equals the proportion of observed values of Z that are less than x in an infinitelly large sample of observations.

This elaboration and explanation have been based on an implicit assumption that the samples that generated the order statistics are independent. The next step is to assume that a variable Z obeys the distribution described by equation (6.7) and recover the results for the order statistics. Should this approach fail to yield the desired results, an attempt will be made to investigate the converse problem under the broader assumption of dependence of the samples.

APPENDIX

PROOF OF THEOREM 1

Theorem 1 is a special case of the following theorem:

Theorem 2 *The class of multiple integrals defined by*

$$\iint_{0<t_1+t_2+\cdots+t_n\leq t}\cdots\int g(t_1+t_2+\cdots+t_{n-s})\phi_1(t_1)\phi_2(t_2)\cdots\phi_n(t_n)dt_1dt_2\cdots dt_n$$

where for $t>0$, $g(t)\in\mathcal{C}$ (i.e. continuous) and $\phi_i(t)\in\mathcal{K}$ (i.e. with at most a finite number of points of discontinuity in every finite value for every finite interval and such that the integral $\int_o^t|\phi_i(u)|du$ has a finite value for every $t>0$), $i=1,2,\ldots,n$, where n is a positive integer, is reducible to the single integral

$$\int_0^t\{g(u)[\phi_1(u)*\phi_2(u)*\cdots*\phi_{n-s}(u)]\}*\phi_{n-s+1}(u)*\cdots*\phi_n(u)du$$

for $s=1,2,\ldots,n-1$ and

$$\int_0^t g(u)[\phi_1(u)*\phi_2(u)*\cdots*\phi_n(u)]du$$

for $s=0$.

Proof. We briefly prove the Theorem as was stated in Chapter 1 for $s=0$. Let

$$I_n=\iint_{t_1+t_2+\cdots+t_n\leq t}\cdots\int f(t_1+t_2+\cdots+t_n)\phi_1(t_1)\phi_2(t_2)\cdots\phi_n(t_n)dt_1dt_2\ldots dt_n$$

Also, let

$$\lambda=t_3+t_4+\cdots+t_n$$

Then

$$I_n = \iint_{t_3+t_4+\cdots+t_n\le t}\cdots\int \phi_3(t_3)\phi_4(t_4)\cdots\phi_n(t_n)$$
$$\iint_{t_1+t_2\le t-\lambda}\cdots\int f(t_1+t_2+\lambda)\phi_1(t_1)\phi_2(t_2)dt_1dt_2\ldots dt_n \tag{6.8}$$

To reduce

$$I = \int_{t_1+t_2\le t-\lambda}\int f(t_1+t_2+\lambda)\phi_1(t_1)\phi_2(t_2)dt_1dt_2$$

let $t_1 = \tau - \omega$ and $t_2 = \omega$; then

$$\begin{aligned} I &= \int_0^{t-\lambda} f(\tau+\lambda)\int_0^{\tau}\phi_1(\tau-\omega)\phi_2(\omega)d\omega d\tau \\ &= \int_0^{t-\lambda} f(\tau+\lambda)[\phi_1(\tau)*\phi_2(\tau)]d\tau \end{aligned} \tag{6.9}$$

Using (6.9) in (6.8) yields

$$\begin{aligned} I_n &= \iint_{t_3+t_4+\cdots+t_n\le t}\cdots\int \phi_3(t_3)\cdots\phi_n(t_n)\int_0^{t-\lambda} f(\tau+\lambda)[\phi_1(\tau)*\phi_2(\tau)]d\tau dt_3\ldots dt_n \\ &= \iint_{\tau+\tau_3+\cdots+t_n\le t}\cdots\int f(\tau+t_3+\cdots+t_n)[\phi_1(\tau)*\phi_2(\tau)]\phi_3(t_3)\cdots\phi_n(t_n)d\tau dt_3\ldots dt_n \end{aligned} \tag{6.10}$$

Applying the previous reduction procedure successively to this last integral, we obtain the final result:

$$I_n = \int_0^t f(\tau)[\phi_1(\tau)*\phi_2(\tau)*\cdots*\phi_n(\tau)]d\tau \tag{6.11}$$

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BIOGRAPHICAL SKETCH

Konstantinos Gakis was born in Serres, Macedonia, Greece, on October 1, 1965. He received his undergraduate degree in Mechanical Engineering from the Faculty of Engineering of the Aristotle University of Thessaloniki, Macedonia, Greece, in July 1988. In November 1988 he became a member of the Technical Chamber of Greece and hence a Lisenced Engineer. In August 1989 he came to the United States and joined the University of Florida, Gainesville, Florida, where he received his Master of Science degree from the Department of Industrial and Systems Engineering. He continued his studies in the same department as a Ph.D. student. Throughout his graduate studies he has been working as a graduate teaching and research assistant.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Boghos D. Sivazlian, Chair
Professor of Industrial and Systems Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

James F. Burns
Professor of Industrial and Systems Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

John F. Mahoney
Professor of Industrial and Systems Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Sencer Yeralan
Associate Professor of Industrial and Systems Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Rocco Ballerini

Rocco Ballerini
Associate Professor of Statistics

This dissertation was submitted to the Graduate Faculty of the College of Engineering and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

August 1993

for Winfred M. Phillips
Dean, College of Engineering

Madelyn M. Lockhart
Dean, Graduate School